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Mathematics

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Chapter 17 and chapter 18

EXPONENTIAL GROWTH AND DECAY *and* ***Surds***

Exponential growth

In Chapter 13, you met compound interest. This involved a constant multiplier and is an example of exponential growth.

Consider \$200 invested at 5% per year compound interest.
After 1 year, it will be worth 200 × 1.05 = \$210. After 2 years, it will be worth

$200 \times 1.05 \times 1.05 = 200 \times 1.05^2 = \$220.50$.

After 3 years, it will be worth

$200 \times 1.05 \times 1.05 \times 1.05 = 200 \times 1.05^3 = \$231.53$.

After 20 years, it will be worth 200×1.05^{20}.

On a calculator, this is done using the powers key. This is usually labelled $\boxed{x^{\blacksquare}}$

The calculation is 200×1.05 $\boxed{x^{\blacksquare}}$ $20 = \$530.66$.

The formula for this calculation is $\$A = 200 \times 1.05^n$, where $\$A$ is the amount of the investment and n is the number of years.

Example 17.1

Question

The number of bacteria present in a population doubles every hour.
If there are 500 bacteria present at 12 noon, find the number present

a at 2 p.m.
b at midnight
c after n hours.

Solution

a $500 \times 2 \times 2 = 2000$
b $500 \times 2^{12} = 2048000$
c 500×2^n

Exponential decay

A population of bats is declining by 15% a year.
At the start, there are 140 bats. 100% − 15% = 85%
After 1 year, there will be 140 × 0.85 = 119

After 2 years, there will be $140 \times 0.85 \times 0.85 = 140 \times 0.85^2 = 101$, and so on.

This calculation, where the constant multiplier is less than 1, is an example of exponential decay.

The calculations involved work just like those for exponential growth, except that the multiplier is less than 1, so the result gets smaller.

After 10 years, there will be $140 \times 0.85^{10} = 28$

The formula for this calculation is $A = 140 \times 0.85^n$, where A is the number of bats and n is the number of years.

Example 17.2

Question

Each year, a car decreases by 12% of its value at the beginning of the year.
At the beginning of the year, a car was valued at \$9000.
Find its value after three years.

Solution

100% − 12% = 88%
Value after three years = $\$9000 \times (0.88)^3 = \$6133.25$ (to the nearest cent).

Note

When the value of something decreases, as in Example 17.2, it is often called *depreciation*.

Key points

- Exponential growth can be calculated using a constant multiplier greater than 1.
- Exponential decay uses a constant multiplier less than 1.

Surds are irrational numbers such as $\sqrt{2}$ or $5+\sqrt{3}$.

Numbers such as $\sqrt{36}$ are not surds, since 36 is a perfect square. $\sqrt{36} = 6$, which is a rational number.

Surds can be expressed in the form $a+b\sqrt{c}$, where a and b are rational numbers and c is an integer that is not a perfect square.

Simplifying surds

Surds can often be simplified by using the result

$\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$.

This result can be demonstrated using $\sqrt{36}$.

$\sqrt{36} = 6 = 2 \times 3 = \sqrt{4} \times \sqrt{9}$

So $\sqrt{36} = \sqrt{4 \times 9} = \sqrt{4} \times \sqrt{9}$

Note

By definition of what we mean by a square root, $\sqrt{a} \times \sqrt{a} = a$.

Example 18.1

Question

Simplify $\sqrt{50}$.

Solution

$\sqrt{50} = \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$

Example 18.2

Question

Simplify $\sqrt{72}$.

Solution

9 is a factor of 72, so $\sqrt{72} = \sqrt{9} \times \sqrt{8} = 3\sqrt{8}$

However, 4 is a factor of 8, so $3\sqrt{8} = 3 \times \sqrt{4} \times \sqrt{2}$

$= 3 \times 2 \times \sqrt{2}$

$= 6\sqrt{2}$

Or, if you spot straight away that 36 is a factor of 72, $\sqrt{72} = \sqrt{36 \times 2}$

$= \sqrt{36} \times \sqrt{2}$

$= 6\sqrt{2}$

Example 18.3

Question

Simplify $\sqrt{12} \times \sqrt{27}$.

Solution

Method 1

$\sqrt{12} = \sqrt{4 \times 3}$
$= \sqrt{4} \times \sqrt{3}$
$= 2 \times \sqrt{3}$

$\sqrt{27} = \sqrt{9 \times 3}$
$= \sqrt{9} \times \sqrt{3}$
$= 3 \times \sqrt{3}$

So $\sqrt{12} \times \sqrt{27} = 2 \times \sqrt{3} \times 3 \times \sqrt{3}$
$= 2 \times 3 \times \sqrt{3} \times \sqrt{3}$
$= 6 \times 3$
$= 18$

Method 2

$\sqrt{12} \times \sqrt{27} = \sqrt{12 \times 27}$
$= \sqrt{324}$
$= 18$

Note

Look for as large a factor of the number as possible which has an exact square root. In Example 18.1, this is 25. In Example 18.2 it is 36.

Manipulation of expressions of the form $a+b\sqrt{c}$

An expression such as $2 + \sqrt{3}$, which is the sum of a rational number and an irrational number, is irrational.

This is because $2 + 1.732\,050\,808\ldots = 3.732\,050\,808\ldots$ which is itself a decimal that goes on forever without recurring, and so is irrational.

Example 18.4

Question

If $x = 5+\sqrt{3}$ and $y = 3-2\sqrt{3}$, simplify these.

a $x+y$ **b** $x-y$ **c** xy

Solution

a $x+y = 5+\sqrt{3}+3-2\sqrt{3}$
$= 5+3+\sqrt{3}-2\sqrt{3}$
$= 8-\sqrt{3}$

b $x-y = 5+\sqrt{3}-(3-2\sqrt{3})$
$= 5-3+\sqrt{3}+2\sqrt{3}$
$= 2+3\sqrt{3}$

These two results illustrate the fact that when adding and subtracting these numbers, you can deal with the rational and irrational parts separately.

c $xy = (5+\sqrt{3})(3-2\sqrt{3}) = 15-10\sqrt{3}+3\sqrt{3}-2\sqrt{3}\times\sqrt{3}$
$= 15-2\times3-10\sqrt{3}+3\sqrt{3}$
$= 9-7\sqrt{3}$

These expressions can be manipulated using the ordinary rules of arithmetic and algebra.

Example 18.5

Question

If $x = 5+\sqrt{2}$ and $y = 3-\sqrt{2}$, simplify these.

a $x+y$ **b** y^2

Solution

a $x+y = 5+\sqrt{2}+3-\sqrt{2}$
$= 5+3+\sqrt{2}-\sqrt{2}$
$= 8$

Note that this result indicates that it is possible for the sum of two irrational numbers to be rational.

b $y^2 = (3-\sqrt{2})^2$
$= (3-\sqrt{2})(3-\sqrt{2})$
$= 9-3\sqrt{2}-3\sqrt{2}+\sqrt{2}\times\sqrt{2}$
$= 9+2-6\sqrt{2}$
$= 11-6\sqrt{2}$

Note that this result is also an application of the algebraic result

$(a+b)^2 = a^2+2ab+b^2$

Rationalizing denominators

When dealing with fractions, it is easier if the denominator is a rational number. For this, you need to manipulate the fraction. This is called rationalising the denominator.

The rules of fractions are used when the denominator is of the form $\sqrt{b}$ or $a\sqrt{b}$. The next example shows the method.

Example 18.6

Question

Rationalise the denominator in these irrational fractions.

a $\frac{5}{\sqrt{2}}$ **b** $\frac{7}{\sqrt{12}}$

Solution

a Multiply the numerator and the denominator by $\sqrt{2}$. By the rules of fractions, since you have multiplied both the numerator and the denominator by the same quantity, you have not changed the value of the number.

This gives $\frac{5\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}} = \frac{5\sqrt{2}}{2}$ and the denominator is now a rational number.

b First, simplify the denominator and then repeat the process in part **a**, this time multiplying by $\sqrt{3}$.

$$\frac{7}{\sqrt{12}} = \frac{7}{\sqrt{4\times3}} = \frac{7}{2\sqrt{3}}$$

$$\frac{7\sqrt{3}}{2\sqrt{3}\times\sqrt{3}} = \frac{7\sqrt{3}}{2\times3}$$

$$= \frac{7\sqrt{3}}{6}$$

Example 18.7

Question

Simplify

a $(5+\sqrt{3})(5-\sqrt{3})$

b $(7-2\sqrt{5})(7+2\sqrt{5})$

Note

If you remember that $(x+y)(x-y) = x^2-y^2$, you can simplify the working in this example.

Solution

a $(5+\sqrt{3})(5-\sqrt{3}) = 25-5\sqrt{3}+5\sqrt{3}-3 = 25-3 = 22$

b $(7-2\sqrt{5})(7+2\sqrt{5}) = 49+14\sqrt{5}-14\sqrt{5}-4\times5 = 49-20 = 29$

When the denominator is of the form $a+b\sqrt{c}$, you need to use the difference of squares as well as the rules of fractions.

Remember that $(x+y)(x-y) = x^2-y^2$.

When the denominator is of the form $a + b\sqrt{c}$, multiply both the numerator and the denominator by $a - b\sqrt{c}$ to rationalise the denominator.

Example 18.8

Question

Rationalise the denominator of the following, and simplify your answer.

a $\frac{6}{\sqrt{7}-2}$ **b** $\frac{1+\sqrt{3}}{5-2\sqrt{3}}$

Solution

a $\frac{6}{\sqrt{7}-2} = \frac{6}{\sqrt{7}-2} \times \frac{\sqrt{7}+2}{\sqrt{7}+2} = \frac{6(\sqrt{7}+2)}{7-4} = \frac{6(\sqrt{7}+2)}{3} = 2(\sqrt{7}+2)$

b $\frac{1+\sqrt{3}}{5-2\sqrt{3}} = \frac{1+\sqrt{3}}{5-2\sqrt{3}} \times \frac{5+2\sqrt{3}}{5+2\sqrt{3}} = \frac{5+2\sqrt{3}+5\sqrt{3}+2\times 3}{25-4\times 3} = \frac{11+7\sqrt{3}}{13}$

Key points

- Surds are irrational numbers, such as $\sqrt{2}$ or $5+\sqrt{3}$.
- Surds can be simplified using $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$.
- Use the normal rules of algebra to add and multiply different surds of the form $a + b\sqrt{c}$.
- To rationalise the denominator of a fraction:
  - when the denominator is of the form $\sqrt{b}$ or $a\sqrt{b}$, multiply the numerator and the denominator by $\sqrt{b}$
  - when the denominator is of the form $a + b\sqrt{c}$, multiply both the numerator and the denominator by $a - b\sqrt{c}$.

Revision questions

1. A town has a population of 45 000.
This population increases exponentially at a rate of 1.6% per year.
Find the population of the town at the end of 5 years.
Give your answer correct to the nearest hundred.

2. The population of a town decreases exponentially at a rate of 1.7% per year. The population now is 250 000.
Calculate the population at the end of 5 years.
Give your answer correct to the nearest hundred.

3. Paula invests \$600 at a rate of r% per year simple interest. At the end of 10 years, the total interest earned is \$90.
Find the value of r.

4. Eric invests an amount in a bank that pays compound interest at a rate of 2.16% per year.
At the end of 5 years, the value of his investment is \$6999.31.
Calculate the amount Eric invests.

5. Jan invests \$800 at a rate of 3% per year simple interest.
Calculate the value of her investment at the end of 4 years.

6. There are 30 000 lions in Africa.
The number of lions in Africa decreases exponentially by 2% each year.
Find the number of lions in Africa after 6 years. Give your answer correct to the nearest hundred.

7. Show that $(6 + 2\sqrt{12})^2 = 12(7 + 4\sqrt{3})$

Show each stage of your working.

8. Show that $\dfrac{\sqrt{20}+\sqrt{80}}{\sqrt{3}}$ can be expressed in the form $\sqrt{a}$ where a is an integer.

Show your working clearly.

9. Show that $\sqrt{45}+\sqrt{20}=5\sqrt{5}$

Show your working clearly.

10. $a = \sqrt{8}+2$

$b = \sqrt{8}-2$

$T = a^2 - b^2$

Work out the value of T.

Give your answer in the form $c\sqrt{2}$ where c is an integer.

11. Show that $\dfrac{2\sqrt{6}}{\sqrt{5}} - \dfrac{\sqrt{3}}{\sqrt{10}}$ can be written in the form $\dfrac{c\sqrt{d}}{10}$

where c and d are integers.

12. Simplify $\sqrt{80} + \sqrt{2\frac{2}{9}}$

Give your answer in the form $\dfrac{a\sqrt{5}}{b}$ where a and b are integers.