Cambridge OL

Mathematics

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Chapter 24 and chapter 25

Inequalities and Sequences

Chapter 24 - Inequalities

Solving inequalities

$a < b$ means 'a is less than b'

$a \leqslant b$ means 'a is less than or equal to b'

$a > b$ means 'a is greater than b'

$a \geqslant b$ means 'a is greater than or equal to b'

Expressions involving these signs are called **inequalities**.

Example 24.1

Question

Find the integer values of x that satisfy each of these inequalities.

a $-3 < x \leqslant -1$ **b** $1 \leqslant x < 4$

Solution

a If $-3 < x \leqslant -1$, then $x = -2$ or -1. Note that -3 is not included, but -1 is.

b If $1 \leqslant x < 4$, then $x = 1$, 2 or 3. Note that 1 is included, but 4 is not.

In equations, if you always do the same thing to both sides, the equation still has the same solution.

The same is usually true for inequalities, but there is one important exception. If you multiply or divide an inequality by a negative number, you must reverse the inequality sign.

Example 24.2

Question

Solve each of these inequalities and show the solution on a number line.

a $3x + 4 < 10$

b $2x - 5 \leqslant 4 - 3x$

c $x + 4 < 3x - 2$

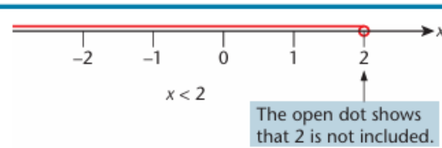
Solution

a

$3x + 4 < 10$

$3x < 6$ Subtract 4 from both sides.

$x < 2$ Divide both sides by 3.



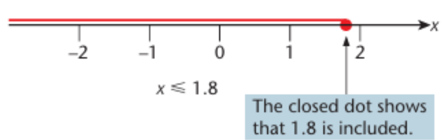
b

$2x - 5 \leqslant 4 - 3x$

$2x \leqslant 9 - 3x$ Add 5 to both sides.

$5x \leqslant 9$ Add $3x$ to both sides.

$x \leqslant 1.8$ Divide both sides by 5.



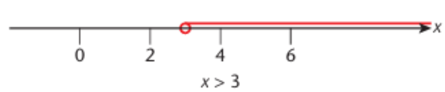
c

$x + 4 < 3x - 2$

$x < 3x - 6$ Subtract 4 from both sides.

$-2x < -6$ Subtract $3x$ from both sides.

$x > 3$ Divide both sides by -2 and change $<$ to $>$ (when dividing by -2).



Example 24.3

Question

Solve this inequality and show the solution on a number line.

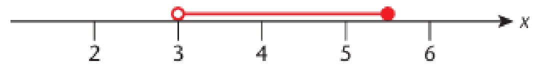
$5 < 2x - 1 \leqslant 10$

Solution

$5 < 2x - 1 \leqslant 10$

$6 < 2x \leqslant 11$ Add 1 to each part of the inequality.

$3 < x \leqslant 5.5$ Divide each part of the inequality by 2.



$3 < x \leqslant 5.5$

Showing regions on graphs

It is often possible to show the region on a graph that satisfies an inequality.

An inequality involving the signs < or > is represented on a graph using a dashed line.

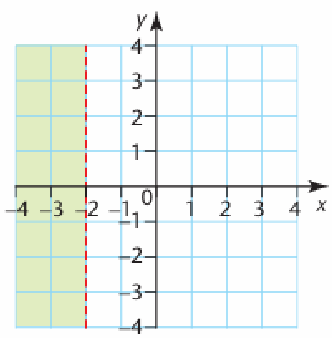
An inequality involving the signs $\leqslant$ or $\geqslant$ is represented by a solid line.

Example 24.4

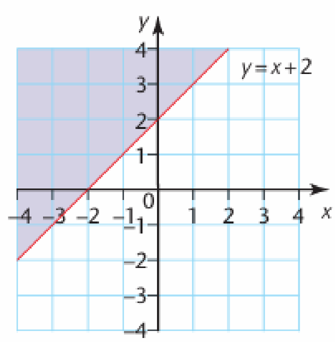
Question

Write down the inequality that describes the region shaded in each graph.

a



b



Solution

a $x < -2$

The line is dashed, so the inequality is either $x < -2$ or $x > -2$.

Choose any point in the shaded region, for example, (–3, 0).

Substitute for x in the inequality.

$-3 < -2$, so the region represents $x < -2$.

b $y \geqslant x + 2$

The line is solid, so the inequality is either $y \geqslant x + 2$ or $y \leqslant x + 2$.

Choose any point in the shaded region, for example, (0, 3).

Substitute for x and y.

$3 \geqslant 0 + 2$, so the region represents $y \geqslant x + 2$

Example 24.5

Question

On separate grids, shade each of these regions.

a $y \geq 2$ **b** $y < 2x - 3$

Solution

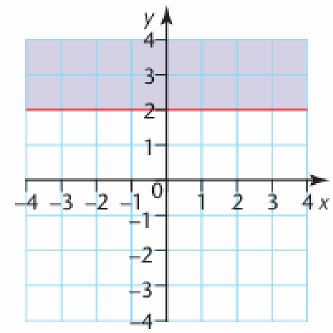
a Draw the solid line $y = 2$.

Choose a point on one side of the line, for example, (0, 0).

Substitute into the inequality.

$0 \geq 2$

Since this is *not* a true statement, (0, 0) is not in the required region.



b Draw the dashed line $y = 2x - 3$.

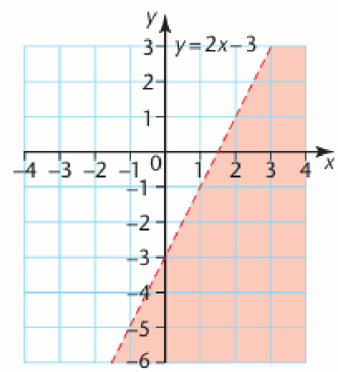
Choose a point on one side of the line, for example, (3, 0).

Substitute into the inequality.

$0 < 2 \times 3 - 3$

$0 < 3$

Since this *is* a true statement, (3, 0) is in the required region.



Note

When testing a point, if possible use (0, 0).

If the line goes through (0, 0), choose a point with positive coordinates, for example (1, 0), to test in the inequality.

Representing regions satisfying more than one inequality

When you represent a region that satisfies more than one inequality, it is often better to shade the region that does not satisfy the inequality; the region that does satisfy the inequality is left unshaded.

Several inequalities can be represented on the same axes and the region where the values of x and y satisfy all of them can be found.

Example 24.6

Question

Draw a set of axes and label them from 0 to 8 for both x and y.

Show, by shading, the region where $x \geq 0$, $y \geq 0$ and $x + 2y \leq 8$.

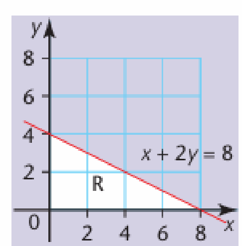
Solution

First, draw the line $x + 2y = 8$.

Then, shade the regions $x < 0$, $y < 0$ and $x + 2y > 8$.

These are the regions where the values of x and y *do not* satisfy the inequalities.

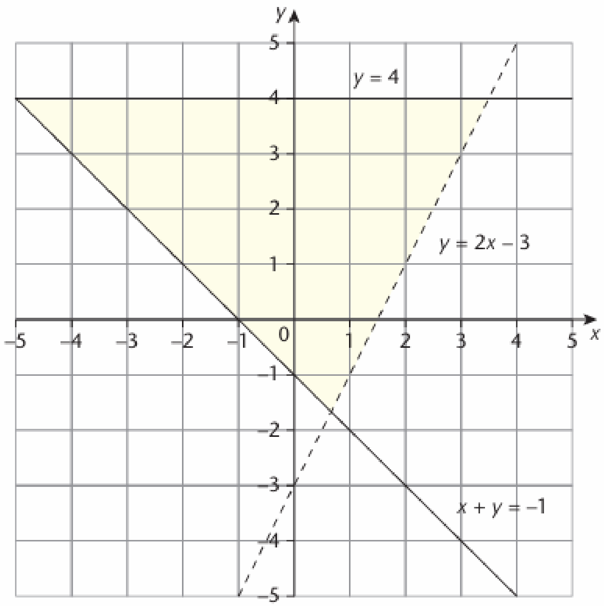
The required region, where the values of x and y satisfy all three inequalities, is left unshaded. It is labelled R.



Example 24.7

Question

List the three inequalities that define the shaded region in the diagram.



Solution

First, choose any point inside the region, such as (0, 0).

Substitute the x- and y-values into each equation to determine the direction of the inequality sign.

$y = 4$	$0 \leqslant 4$	solid line, so equals with the inequality	so,	$y \leqslant 4$
$y = 2x - 3$	$0 > 0 - 3$	dotted line, so no equals with the inequality	so,	$y > 2x - 3$
$x + y = -1$	$0 + 0 \geqslant -1$	solid line, so equals with the inequality	so,	$x + y \geqslant -1$

The three inequalities are $y \leqslant 4$, $y > 2x - 3$, $x + y \geqslant -1$

Key points

- '$a < b$' means 'a is less than b'.
  '$a \leqslant b$' means 'a is less than or equal to b'.
  '$a > b$' means 'a is greater than b'.
  '$a \geqslant b$' means 'a is greater than or equal to b'.
- Inequalities can be treated like equations, except when multiplying or dividing by a negative number. Then, you must reverse the symbol, since, for example, $3 < 5$ but $-3 > -5$.
- An inequality in y and/or x, involving $\leqslant$ or $\geqslant$ is represented by a region on a coordinate grid bounded by a solid line.
- An inequality in y and/or x, involving $<$ or $>$ is represented by a region on a coordinate grid bounded by a dashed line.
- Given an inequality, the region it represents can be found by:
  - drawing the appropriate straight line, solid or dashed
  - using a coordinate point on one side of the line to determine which is the required region.
- Given a region on a coordinate grid, the inequality it represents can be found by:
  - finding the equation of the line which is the boundary
  - using a coordinate point on one side of the line to determine the direction of the inequality
  - including an equals sign in the inequality if the line is solid.
- Given more than one inequality, the region which satisfies them all can be found by shading the region **not** satisfied by each inequality. The part of the grid left unshaded is the required region.
- Given a region on a coordinate grid bounded by straight lines, the inequalities it represents can be found by finding the inequality represented by each line separately.

Chapter 25 – Sequences

Number patterns

Here is a number pattern.

6 11 16 21 26 31 …

You can see that, to get from one number to the next, you add 5.

Here is another number pattern.

1 2 4 8 16 32 …

To get from one number to the next this time you multiply by 2.

$$1 \xrightarrow{\times 2} 2 \xrightarrow{\times 2} 4 \xrightarrow{\times 2} 8 \xrightarrow{\times 2} 16 \xrightarrow{\times 2} 32 \quad \ldots$$

Number patterns like these are called **sequences**.

The rule for finding the next number in a sequence from the previous one is called the **term-to-term rule**.

Example 25.1

Question

Find the term-to-term rule for each of these sequences and give the next number.

a 1 4 7 10 13 16 …

b 63 56 49 42 35 28 …

c 1 000 000 100 000 10 000 1000 100 …

Solution

a To get from one number to the next you add 3.

$$1 \xrightarrow{+3} 4 \xrightarrow{+3} 7 \xrightarrow{+3} 10 \xrightarrow{+3} 13 \xrightarrow{+3} 16 \xrightarrow{+3} \ldots$$

The next number is 16 + 3 = 19.

b To get from one number to the next you subtract 7.

$$63 \xrightarrow{-7} 56 \xrightarrow{-7} 49 \xrightarrow{-7} 42 \xrightarrow{-7} 35 \xrightarrow{-7} 28 \xrightarrow{-7} \ldots$$

The next number is 28 – 7 = 21.

c To get from one number to the next you divide by 10.

$$1\,000\,000 \xrightarrow{\div 10} 100\,000 \xrightarrow{\div 10} 10\,000 \xrightarrow{\div 10} 1000 \xrightarrow{\div 10} 100 \xrightarrow{\div 10} \ldots$$

The next number is 100 ÷ 10 = 10.

In the sequences in the next example, some of the numbers are missing.

It is possible to work out what they should be by looking at the numbers in the sequence and working out how to get from one number to the next.

Example 25.2

Question

Find the missing numbers in each of these sequences.

a 10 17 ☐ ☐ 38 45

b 35 ☐ 23 17 ☐ 5

c ☐ 15 ☐ 23 27 31

Solution

a The term-to-term rule is 'add 7'.

$$10 \xrightarrow{+7} 17 \xrightarrow{+7} \boxed{} \xrightarrow{+7} \boxed{} \xrightarrow{+7} 38 \xrightarrow{+7} 45$$

So the first missing number is 17 + 7 = 24.

The second missing number is 24 + 7 = 31.

b The term-to-term rule is 'subtract 6'.

$$35 \xrightarrow{-6} \boxed{} \xrightarrow{-6} 23 \xrightarrow{-6} 17 \xrightarrow{-6} \boxed{} \xrightarrow{-6} 5$$

So the first missing number is 35 – 6 = 29.

The second missing number is 17 – 6 = 11.

c The term-to-term rule is 'add 4'.

$$\boxed{} \xrightarrow{+4} 15 \xrightarrow{+4} \boxed{} \xrightarrow{+4} 23 \xrightarrow{+4} 27 \xrightarrow{+4} 31$$

To get to the previous number, you must subtract 4.

So the first missing number is 15 – 4 = 11.

You find the second missing number by adding 4 in the usual way.

The second missing number is 15 + 4 = 19.

Linear sequences

Look at this sequence.

2, 5, 8, 11, 14, ...

The terms of the sequence increase by 3 each time.

When the increase is constant like this, the sequence is called a linear sequence.

In the same way, the sequence 11, 9, 7, 5, 3, 1, −1, ... is also linear, since the terms decrease by the same amount (2) each time.

Another way of thinking of this is that you are adding −2 each time.

This idea will help with finding the *n*th term.

The nth term

It is often possible to find a formula to give the terms of a sequence.

Example 25.3

Question

The formula for the *n*th term of a sequence is $2n + 1$. Find each of these terms.

a u_1 **b** u_2 **c** u_3

Solution

a $u_1 = 2 \times 1 + 1 = 3$

b $u_2 = 2 \times 2 + 1 = 5$

c $u_3 = 2 \times 3 + 1 = 7$

Note

You usually use n to stand for the number of a term and u_n for the nth term.

Finding the formula for the nth term

Look back at the formulas in Exercise 25.3 questions 1–20 and the sequences in your answers and notice how they are connected.

All the sequences in questions 1–20 are linear. If the formula contains the expression 2n, the terms increase by 2 each time. If it contains the expression 5n, the terms increase by 5 each time.

Note

This will always work if the differences are constant.

The difference between terms in a linear sequence is always constant.

Similarly, if it contains the expression −3n, the terms increase by −3 (or decrease by 3) each time.

So, to find a formula for the nth term of a given sequence, find how much more (or less) each term is than the one before it.

Example 25.4

Question

Find the formula for the *n*th term of this sequence.
3, 5, 7, 9, ...

Solution

The difference between each term is 2, so the formula will include the expression $2n$.

When $n = 1$, $2n = 2$.

But the first term of the sequence is 3, which is 1 more.

So the formula for the *n*th term of this sequence is $2n + 1$.

You can also use algebra to find the formula for the nth term. The nth term can be written as T_n or u_n.

Example 25.5

Question

Find the formula for the nth term of this sequence.

20, 17, 14, 11, …

Solution

As the difference between the terms is constant, this is a linear sequence.

The general formula for the nth term of a linear sequence is

$T_n = an + b$.

$T_1 = 20$ so $20 = a + b$

$T_2 = 17$ so $17 = 2a + b$

Solving the simultaneous equations gives

$a = -3$ and $b = 23$

So $T_n = 23 - 3n$

Note

Notice the similarity between this and the equation of a straight line, $y = mx + c$.

Note

Check that your formula for the nth term is correct by trying it out for the first few terms. Substitute $n = 1, 2, 3, \ldots$, into your formula.

The nth term of quadratic, cubic and exponential sequences

This is the sequence of square numbers.

1^2	2^2	3^2	4^2	5^2	…	n^2	…
↓	↓	↓	↓	↓		↓	
1	4	9	16	25	…	n^2	…

This is the sequence of cube numbers.

1^3	2^3	3^3	4^3	5^3	…	n^3	…
↓	↓	↓	↓	↓	…	↓	…
1	8	27	64	125	…	n^3	…

This is an exponential sequence.

3^1	3^2	3^3	3^4	3^5	…	3^n
↓	↓	↓	↓	↓	…	↓
3	9	27	81	243	…	3^n

You can find the formula for the nth term of a quadratic, cubic or exponential sequence by comparing the sequence with the sequence of square numbers, cube numbers or exponential numbers.

Example 25.6

Question

Find the formula for the nth term of this cubic sequence.

$-1 \quad 6 \quad 25 \quad 62 \quad 123 \quad \ldots$

Solution

Compare the given sequence with the sequence of cube numbers.

1	8	27	64	125	…	n^3
↓−2	↓−2	↓−2	↓−2	↓−2	…	↓−2
−1	6	25	62	123	…	$n^3 - 2$

Notice that every term is 2 smaller than the corresponding term in the sequence of cube numbers.

So the formula is $n^3 - 2$.

Example 25.7

Question

Find the formula for the nth term of this quadratic sequence.

$8 \quad 13 \quad 20 \quad 29 \quad 40 \quad \ldots$

Solution

Since this is a quadratic sequence, use $T_n = an^2 + bn + c$.

$n = 1 \qquad 8 = a + b + c$

$n = 2 \qquad 13 = 4a + 2b + c \qquad 5 = 3a + b$

$n = 3 \qquad 20 = 9a + 3b + c \qquad 7 = 5a + b \qquad 2 = 2a$

$a = 1$

Substitute $a = 1 \qquad 5 = 3 + b$

$b = 2$

Substitute $a = 1$, $b = 2$

$8 = 1 + 2 + c$

$c = 5$

So that $T_n = n^2 + 2n + 5$

Example 25.8

Question

The nth term of a sequence is found using the formula $u_n = An^2 + B^n$.

$u_1 = 3$ and $u_2 = 8$.

Find u_3.

Solution

Substitute for $n = 1$ and $n = 2$ in the formula.

$3 = A + B$ (1)

$8 = 4A + B^2$ (2)

Solve the simultaneous equations.

$12 = 4A + 4B$ (1) × 4

$8 = 4A + B^2$

$4 = 4B - B^2$ Subtract.

$B^2 - 4B + 4 = 0$ Solve the quadratic equation.

$(B - 2)^2 = 0$

So $B = 2$

Substituting in equation (1) gives $A = 1$.

So $u_n = n^2 + 2^n$

$u_3 = 3^2 + 2^3 = 9 + 8 = 17$

Note

Remember to check your answer by substituting in the formula.

Key points

- The term-to-term rule for a sequence of numbers can be used to find the terms of a sequence.
- A linear sequence increases or decreases by a fixed value.
- The formula for the nth term of a sequence can be used to find every term in the sequence by substituting values for n. The first term is found using $n = 1$, the second term using $n = 2$, and so on.
- The formula for the nth term of a linear sequence has the form $an + b$, where a is the fixed value by which the sequence is changing from term to term and b is an integer to be found.
- Other sequences include quadratic (for example, n^2), cubic (for example, n^3) and exponential (for example, 2^n) sequences.
- The formula for the nth term of a non-linear sequence can be found by using a general formula for that sequence. For a quadratic sequence, use $T_n = an^2 + bn + c$ and substitute $n = 1$ and T_1, $n = 2$ and T_2, $n = 3$ and T_3. Solve the simultaneous equations to find a, b and c.

Revision questions

1. Find the integer values that satisfy the inequality $2 < 2x \leq 10$.

2. Solve the inequality $n + 7 < 5n - 8$.

3. Solve the inequality $3m + 12 \leq 8m - 5$.

4. Solve the inequality $3(x-2) < 7(x+2)$.

5. In a sequence

$$T_1 = 17 \quad T_2 = 12 \quad T_3 = 7 \quad T_4 = 2.$$

Find

T_5

6. The nth term of a sequence is $60 - 8n$.

Find the largest number in this sequence.

7. Here is a sequence of numbers.

$$7,\ 5,\ 3,\ 1,\ -1, \dots$$

Find an expression for the n th term of this sequence.

8.

$\frac{1}{3}, \frac{3}{4}, \frac{4}{7}, \frac{7}{11}, \frac{11}{18}, \ldots$

i) One term of this sequence is $\frac{p}{q}$.

Find, in terms of p and q, the next term in this sequence.

ii) Find the 6th term of this sequence.

9.

These are the first five terms of a sequence.

$\frac{1}{2} \qquad \frac{3}{4} \qquad \frac{7}{6} \qquad \frac{13}{8} \qquad \frac{21}{10}$

Find the next term.

10.

The nth term of a sequence is given by $an^2 + bn$ where a and b are integers.

The 2nd term of the sequence is −2
The 4th term of the sequence is 12

Find the 6th term of the sequence.

11.

Sequence	1st term	2nd term	3rd term	4th term	5th term		nth term
A	13	9	5	1			
B	0	7	26	63			
C	$\frac{7}{8}$	$\frac{8}{16}$	$\frac{9}{32}$	$\frac{10}{64}$			

Complete the table for the three sequences.

12.

The table shows the first five terms of sequence *A* and sequence *B*.

Term	1	2	3	4	5	6
Sequence *A*	7	13	23	37	55	
Sequence *B*	1	3	9	27	81	

Complete the table for the 6th term of each sequence.

13.

The first four terms u_1, u_2, u_3, and u_4 in a sequence of numbers are given by

$$u_1 = \mathbf{1} \times 2 + 3^2 = 11$$
$$u_2 = \mathbf{2} \times 3 + 4^2 = 22$$
$$u_3 = \mathbf{3} \times 4 + 5^2 = 37$$
$$u_4 = \mathbf{4} \times 5 + 6^2 = 56.$$

Evaluate u_5.

14.

A sequence has nth term $2n^2 + 5n - 15$.

Find the difference between the 4th term and the 5th term of this sequence.