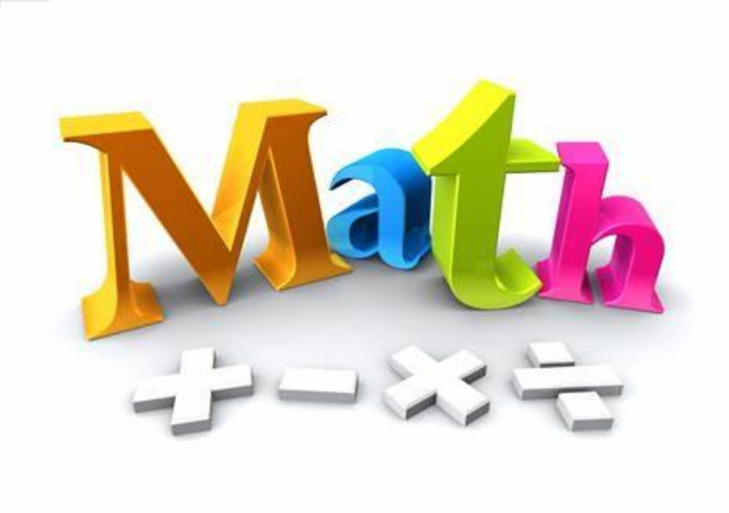
Cambridge OL

Mathematics

CODE: (4024)

Chapter 27 and Chapter 28

Graphs in practical situations and graphs of functions



Conversion graphs

A conversion graph can be used to convert quantities from one unit to another. They are often used to convert between different currencies.

Note

Conversion graphs will only give approximate values. If you know the conversion rate, then you can find a more accurate answer by calculation.

Example 27.1

Question

The exchange rate between pounds and dollars is £1 = $1.60.

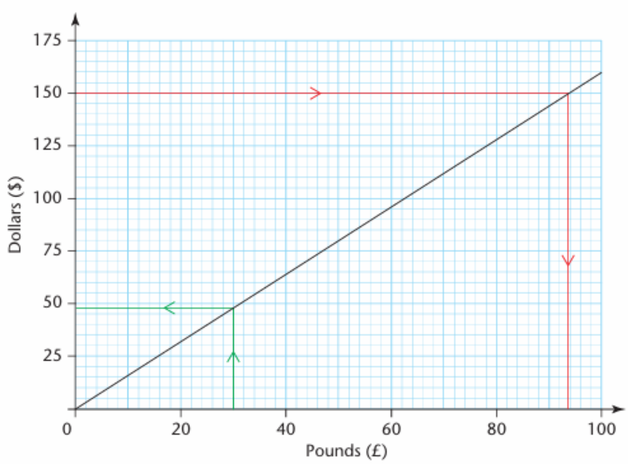
a Draw a conversion graph for pounds to dollars.

b Use your graph to convert

i £30 to dollars **ii** $150 to pounds.

Solution

a As £1 is equivalent to $1.60, then £100 is equivalent to $160.
Draw a straight line from (0, 0) to (100, 160).
Label each axis with the currency.



b i Reading from the graph, £30 = $48

This is an estimated value.

The line meets the axis between $45 and $50, closer to $50 so it is estimated at $48.

ii Reading from the graph, $150 = £94

This is an estimated value.

The line meets the axis between £92 and £94, but very close to £94, so the answer is given correct to the nearest pound.

Example 27.2

Question

Lalith ran the first 3 kilometres to school at a speed of 12 km/h.

She then waited 5 minutes for her friend.

They walked the last 2 kilometres to school together, taking 20 minutes.

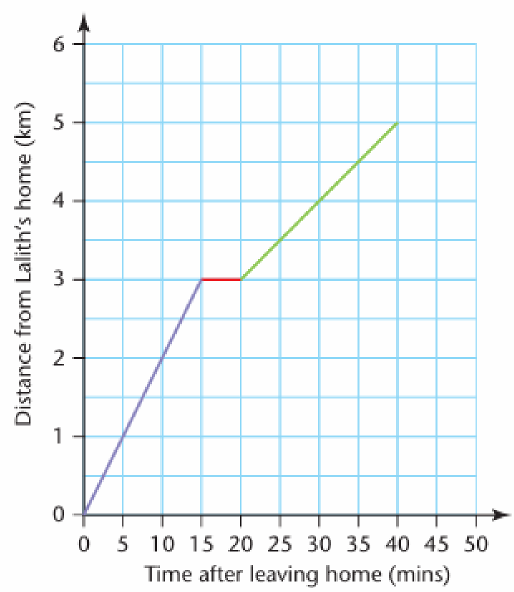
Draw a graph to show Lalith's journey.

Solution

It takes 15 minutes to run 3 kilometres at 12 km/h, so the first part of the graph is a straight line from the origin to (15, 3).

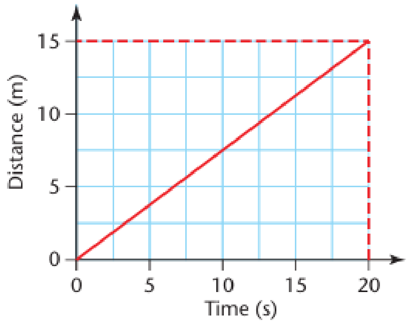
The second part of the graph is horizontal from (15, 3) to (20, 3) as Lalith is not moving for 5 minutes.

The final part of the journey covers a further 2 kilometres in 20 minutes, so is a straight line from (20, 3) to (40, 5). This assumes that they walk at a constant speed.



Rate of change on a distance–time graph

The gradient of a graph is a measure of its steepness or rate of change.



For this distance–time graph

$$\text{gradient} = \frac{\text{change in distance}}{\text{change in time}}$$

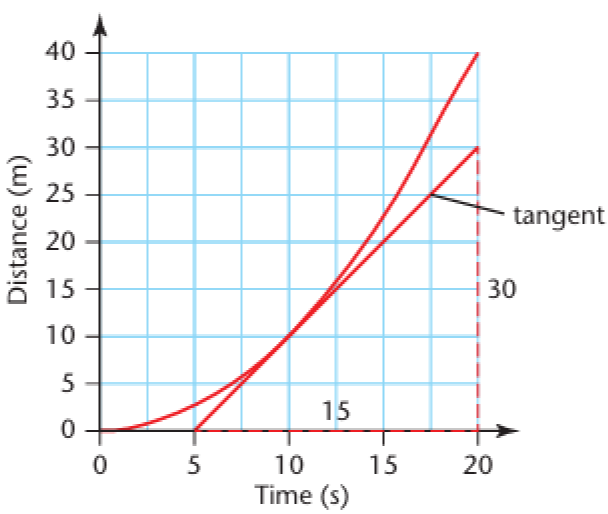
$$= \frac{15\text{ metres}}{20\text{ secounds}}$$

$$= 0.75\text{ metres per second}$$

The units of the gradient in this case are metres per second or m/s. The gradient gives the speed of the object.

The graph is a straight line, so the object is moving with a constant speed of 0.75 m/s.

If the distance–time graph is not a straight line, then the speed of the object is changing.



The graph is curving upwards, so the speed is increasing. You can estimate the speed of the object at a given time by drawing a line which touches the curve at that point. This is called a tangent.

You then calculate the gradient of the tangent to the curve at that point

The gradient of the tangent to the curve at time 10 seconds gives the speed of the object after 10 seconds.

$$\text{gradient} = \frac{\text{change in distance}}{\text{change in time}}$$

$$= \frac{30 \text{ metres}}{15 \text{ seconds}}$$

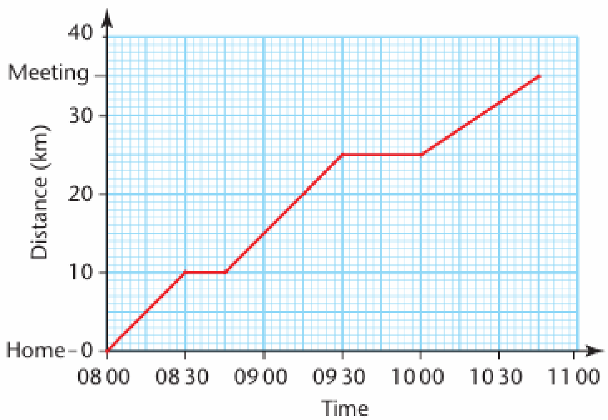
$$= 2 \text{ metres per second}$$

The speed of the object after 10 seconds is 2 m/s.

Example 27.3

Question

This distance–time graph shows Manjit's journey from his home to a meeting.



a Work out Manjit's speed, in kilometres per hour, for each stage of the journey.

b Sketch a speed–time graph for Manjit's journey.

Solution

a The graph is made up of five straight-line segments.

Each of these lines represents a stage with a constant speed.

To work out the speed, find the gradient of the line.

From 08 00 to 08 30, speed $= \frac{10\text{ km}}{0.5\text{ hours}} = 20\text{ km/h}$

From 08 30 to 08 45, the line is horizontal, so Manjit is not moving.

From 08 45 to 09 30, speed $= \frac{(25-10)\text{km}}{0.75\text{ hours}} = 20\text{ km/h}$

From 09 30 to 10 00, the line is horizontal, so Manjit is not moving.

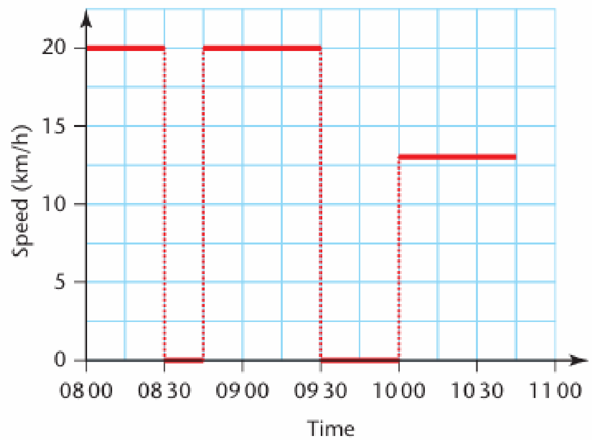
From 10 00 to 10 45, speed $= \frac{(35-25)\text{km}}{0.75\text{ hours}} = 13.33\ldots\text{ km/h}$

Note

The question asks for the speed in kilometres per hour so you must use the correct units in the gradient calculation.

You must convert 30 minutes to 0.5 hours and 45 minutes to 0.75 hours.

b The speed–time graph is made up of five horizontal line segments.

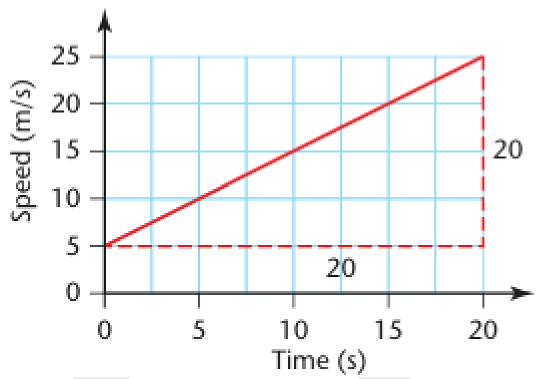


Note

This is a sketch of the situation, because the actual speed would not change instantaneously from 20 km/h to 0 km/h, for example.

Rate of change on a speed–time graph

This speed–time graph shows the speed of an object during a given time period.



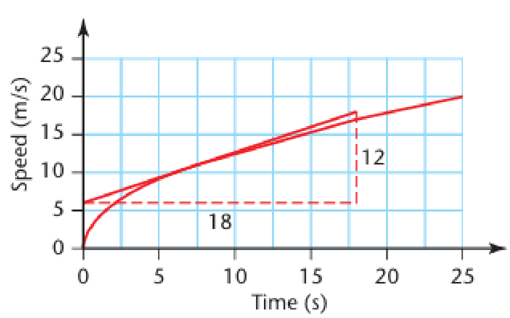
The gradient of the graph represents the rate of change of speed with time. The graph is a straight line, so the speed is changing at a constant rate.

$$\text{gradient} = \frac{20 \text{ m/s}}{20 \text{ s}} = 1 \text{ m/s}^2$$

The rate of change of speed with time is the **acceleration** of the object.

The units of acceleration in this case are metres per second squared or m/s^2.

If the speed–time graph is not a straight line, then the acceleration is changing.



The graph is curving downwards, so the acceleration is decreasing. You can estimate the acceleration of the object at a given time by calculating the gradient of the tangent to the curve at that point.

The gradient of the tangent to the curve at time 7.5 seconds gives the speed of the object after 7.5 seconds.

$$\text{gradient} = \frac{\text{change in speed}}{\text{change in time}}$$

$$= \frac{12\,\text{m/s}}{18\,\text{seconds}}$$

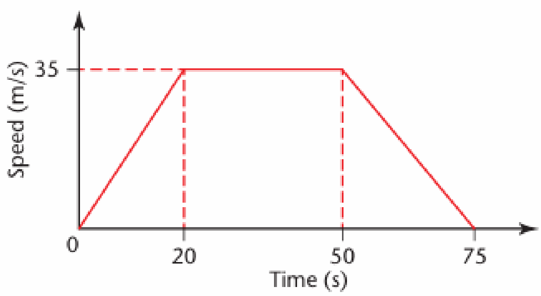
$$= 0.67\,\text{m/s}^2 \text{ correct to 2 decimal places}$$

The acceleration of the object after 7.5 seconds is $0.67\,\text{m/s}^2$.

Example 27.4

Question

The speed–time graph shows part of a car's journey.



a Work out the acceleration, in m/s^2, for each stage of this journey.

b Calculate the speed of the car after 60 seconds.

Solution

a The graph is made up of three straight-line segments.
Each of these lines represents a stage with constant acceleration.
To work out the acceleration, find the gradient of the line.

From 0 to 20 seconds, acceleration $= \frac{35\,\text{m/s}}{20\,\text{s}} = 1.75\,\text{m/s}^2$.

From 20 to 50 seconds, the speed is constant, so acceleration $= 0\,\text{m/s}^2$.

From 50 to 75 seconds, acceleration $= \frac{-35\,\text{m/s}}{25\,\text{s}} = -1.4\,\text{m/s}^2$.

b You know that the speed of the car is 35 m/s after 50 seconds.
After this the car is decelerating at $1.4\,\text{m/s}^2$.
In 10 seconds, the speed will decrease by $10 \times 1.4 = 14\,\text{m/s}$.
After 60 seconds, the speed is $35 - 14 = 21\,\text{m/s}$.

Note

When the acceleration is negative, the car is slowing down.

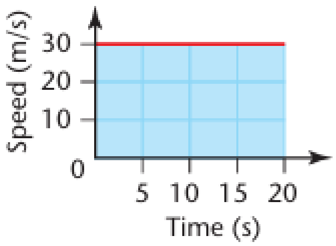
Negative acceleration is known as **deceleration**.

You can say that from 50 to 75 seconds, the car has a deceleration of $1.4\,\text{m/s}^2$.

Area under a speed–time graph

The distance travelled by an object moving at 30 m/s for 20 seconds can be found using the formula distance = speed × time. Distance = 30 × 20 = 600 metres

This is the speed–time graph for an object moving with a constant speed of 30 m/s for 20 seconds.



The blue rectangle is the area under the graph. The area of the rectangle is 30 × 20 = 600.

The units of the area in this case are metres per second × seconds = metres.

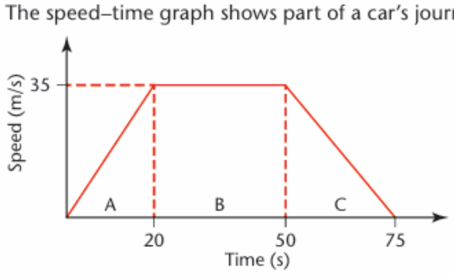
So the area under the graph represents 600 metres. This is the distance travelled by the object.

The area under a speed–time graph gives the distance travelled by the object in that time.

Example 27.5

Question

The speed–time graph shows part of a car's journey.



Calculate the distance travelled by the car in these 75 seconds.

Solution

The area under the graph gives the distance travelled.

The area can be found by adding the areas of two triangles and a rectangle.

Distance travelled = area of triangle A + area of rectangle B + area of triangle C

$= \frac{1}{2} \times 20 \times 35 + 30 \times 35 + \frac{1}{2} \times 25 \times 35$

= 1837.5 metres

Key points

- Conversion graphs can be used to convert from one unit to another.
- Travel graphs represent distance against time and show a journey. The rate of change or slope of the line represents the speed. If the line is not straight, the rate of change can be estimated by drawing a tangent at the point.
- The rate of change on a speed–time graph represents the acceleration.
- The area under a speed–time graph gives the distance travelled.

28. GRAPHS OF FUNCTIONS

Quadratic graphs

Quadratic graphs are graphs of equations of the form $y = ax^2 + bx + c$, where a, b and c are constants, and b and c may be zero.

When the graph of a quadratic equation is drawn, it produces a curve called a **parabola**.

Example 28.1

Question

a Draw the graph of $y = x^2 - 2x - 3$ for values of x from –2 to 4.

Label the axes from –2 to 4 for x and from –5 to 5 for y.

b Find the values of x for which $y = 0$.

Note

Sometimes a table will be given with only the two rows for the x and y values.

You may find it useful to include all the rows and then just add the correct rows to get the values of y.

Solution

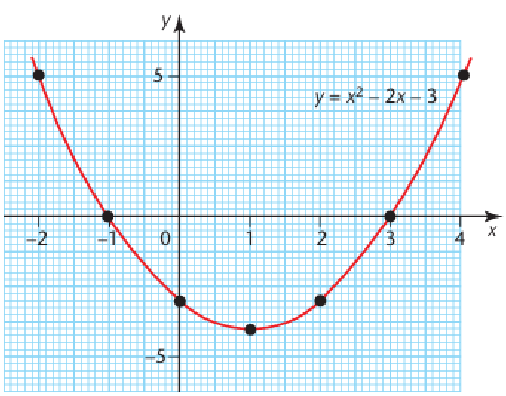
a

x	–2	–1	0	1	2	3	4
x^2	4	1	0	1	4	9	16
$-2x$	4	2	0	–2	–4	–6	–8
-3	–3	–3	–3	–3	–3	–3	–3
$y = x^2 - 2x - 3$	5	0	–3	–4	–3	0	5

To find the value of y, add together the values in rows 2, 3 and 4.

Now plot the points (–2, 5), (–1, 0), and so on. Join them with a smooth curve.

Label the curve.



Note

Practise drawing smooth curves. Keep your pencil on the paper. Make the curve a thin single line. Look ahead to the next point as you draw the line. Keep your hand inside the curve by turning the paper.

b $y = 0$ when the curve crosses the x-axis.

This is when $x = -1$ or $x = 3$.

Note

Complete a table of values to work out the points.

Make your own table if needed.

A common error is to include the value in the x-row when adding to find y, so separate this row off clearly.

Note

If you find that you have two equal lowest (or highest) values for y, the curve will go below (or above) that value.

To find the y-value between the equal values, find the value of x halfway between the two points. Substitute to find the corresponding value of y.

Note

In question 2, to find the value of y, rows 3 and 4 are added.

Note

An extra point at $x = 2.5$ might be useful for question 7.

Note

Note that in some questions, the values of y are not in pairs, so the lowest point is not exactly halfway between two of the given points.

Using graphs to solve equations

One way of finding solutions to quadratic equations is to draw and use a graph.

Example 28.2

Question

a Draw the graph of $y = x^2 - 2x - 8$ for values of x from –3 to 5.

b Solve the equation $x^2 - 2x - 8 = 0$.

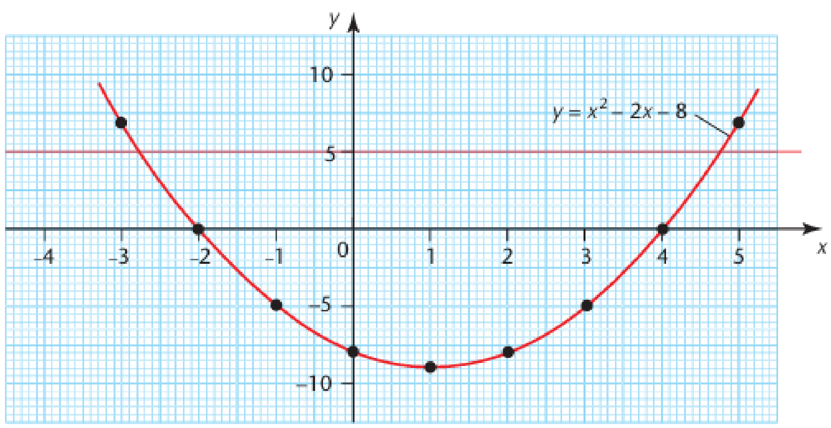
c Solve the equation $x^2 - 2x - 8 = 5$.

Note
The solutions of an equation are also called the roots of the equation.

Solution

a

x	–3	–2	–1	0	1	2	3	4	5
x^2	9	4	1	0	1	4	9	16	25
$-2x$	6	4	2	0	–2	–4	–6	–8	–10
-8	–8	–8	–8	–8	–8	–8	–8	–8	–8
$y = x^2 - 2x - 8$	7	0	–5	–8	–9	–8	–5	0	7



b The solution of $x^2 - 2x - 8 = 0$ is when $y = 0$, where the curve cuts the x-axis. The solution is $x = -2$ or $x = 4$.

c The solution of $x^2 - 2x - 8 = 5$ is when $y = 5$.

Draw the line $y = 5$ on your graph and read off the values of x where the curve cuts the line.

The solution is $x = -2.7$ or $x = 4.7$, to 1 decimal place.

Sometimes you may have drawn a quadratic graph and then need to solve an equation that is different from the graph you have drawn. Rather than drawing another graph, it may be possible to rearrange the equation to obtain the one you have drawn.

Example 28.3

Question

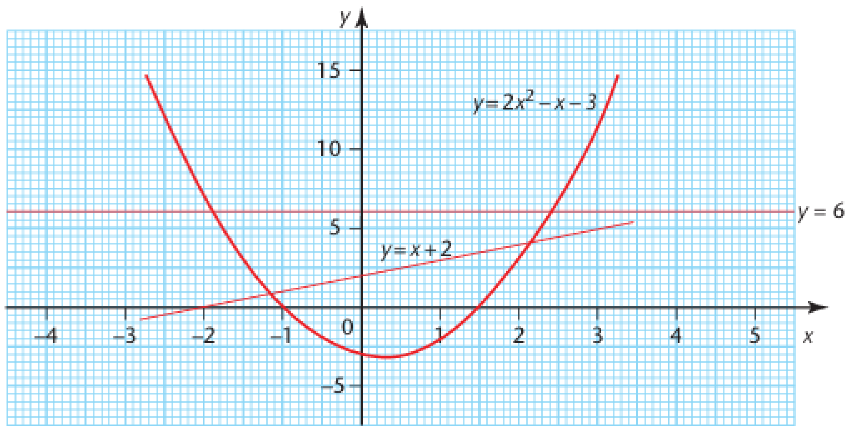
a Draw the graph of $y = 2x^2 - x - 3$ for values of x between –3 and 4.

b Use this graph to solve these equations.

i $2x^2 - x - 3 = 6$ **ii** $2x^2 - x = x + 5$

Solution

a



b **i** To solve $2x^2 - x - 3 = 6$ you draw the line $y = 6$ on your graph.

At the points of intersection of the line and the curve, $y = 6$ and $2x^2 - x - 3$ and therefore $2x^2 - x - 3 = 6$.

The curve and line cross at approximately $x = -1.9$ and $x = 2.4$.

ii To solve $2x^2 - x = x + 5$ you must rearrange the equation so that you have $2x^2 - x - 3$ on the left-hand side.

You do this by subtracting 3 from both sides.

$2x^2 - x = x + 5$

$2x^2 - x - 3 = x + 2$

Now draw the line $y = x + 2$ on your graph.

The points where the line and the curve intersect are the solution to $2x^2 - x = x + 5$.

The curve and line cross at approximately $x = -1.2$ and $x = 2.2$.

Cubic graphs

Cubic graphs are graphs of equations of the form

$$y = ax^3 + bx^2 + cx + d$$

where a, b, c and d are constants, and b, c and d may be zero.

Example 28.4

Question

a Draw the graph of $y = x^3$ for values of x from –3 to 3.
Label the x-axis from –3 to 3 and the y-axis from –30 to 30.

b Use your graph to solve the equation $x^3 = 12$. Give your answer to 1 decimal place.

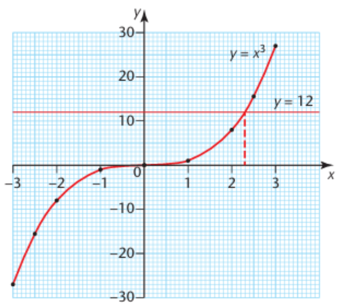
Solution

a

x	−3	−2.5	−2	−1	0	1	2	2.5	3
$y = x^3$	−27	−15.625	−8	−1	0	1	8	15.625	27

The outside points are a long way apart, so plotting the values for $x = 2.5$ and $x = -2.5$ helps with the drawing of the curve.

b To solve $x^3 = 12$, you need to draw the line $y = 12$ on your graph and find where it intersects the curve.
The solution is $x = 2.3$, to 1 decimal place.



The curve goes from bottom left to top right and has a 'double bend' in the middle.

All cubic curves with a positive x3 term have a similar shape.

If the x^3 term is negative the curve goes from top left to bottom right.

Example 28.5

Question

a Draw the graph of $y = x^3 - 2x$ for values of x from –2 to 2.
Label the x-axis from –2 to 2 and the y-axis from –4 to 4.

b Use the graph to find the roots of the equation $x^3 - 2x = 0$.
Give your answers correct to 1 decimal place.

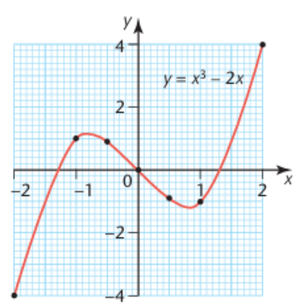
Solution

a

x	−2	−1	−0.5	0	0.5	1	2
x^3	−8	−1	−0.125	0	0.125	1	8
$-2x$	4	2	1	0	−1	−2	−4
$y = x^3 - 2x$	−4	1	0.875	0	−0.875	−1	4

It helps to see more clearly where the curve is highest and lowest if you work out the values of y for $x = -0.5$ and $x = 0.5$.

b To find the roots of $x^3 - 2x = 0$, you need to find where $y = x^3 - 2x$ crosses $y = 0$, which is the x-axis.
The roots are $x = -1.4$ (to 1 d.p.), 0 and 1.4 (to 1 d.p.).



The shape of the curve is similar to the previous one. The −2x term makes the 'double bend' more pronounced.

Reciprocal graphs

Reciprocal graphs are graphs of equations of the form

$y = \frac{a}{x}$

where a is a non-zero constant.

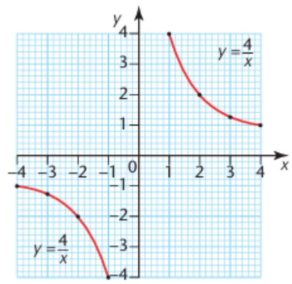
Example 28.6

Question

Draw the graph of $y = \frac{4}{x}$.

Solution

x	−4	−3	−2	−1.5	−1	1	1.5	2	3	4
$y = \frac{4}{x}$	−1	−1.33…	−2	−2.66…	−4	4	2.66…	2	1.33…	1



You cannot use 0 as a point in this type of graph, since you cannot divide 4 (or any other number) by 0.

Again, it is helpful to work out extra points to give a better curve. In this case you might work out the value of y when $x = -1.5$ and 1.5.

It is also useful to have the same scale for both x and y in this case.

All equations of this type have graphs of the same shape, with two separate branches in opposite quadrants.

Plotting points for x = 0.5 and x = 0.1 would help to show that the curve gets closer to the y-axis without ever meeting it.

Plotting points for x = 5 and x = 6 would help to show that the curve gets closer to the x-axis without ever meeting it

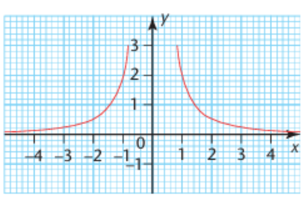
Example 28.7

Question

Draw the graph of $y = \frac{2}{x^2}$.

Solution

x	−4	−3	−2	−1.5	−1	1	1.5	2	3	4
$y = \frac{2}{x^2}$	0.125	0.22…	0.5	0.88…	2	2	0.88…	0.5	0.22…	0.125



All equations of this type have graphs of the same shape, with two separate branches side by side.

Plotting points for x = 0.5 and x = 0.1 would help to show that the curve gets closer to the y-axis without ever meeting it.

Plotting points for x = 5 and x = 6 would help to show that the curve gets closer to the x-axis without ever meeting it.

Other graphs

In reciprocal graphs, you have learned that x cannot be zero. Here is another type of graph where the value of x is restricted:

$y = a\sqrt{x}$, or $y = ax^{\frac{1}{2}}$, where a is a number.

Here, since the square of a number cannot be negative, x cannot be negative.

Example 28.8

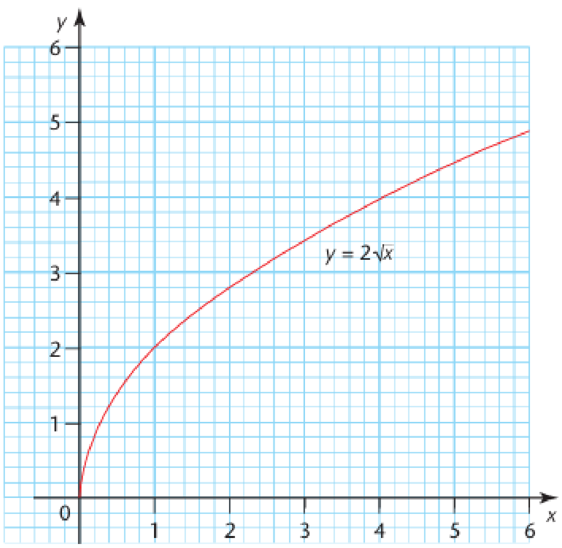
Question

Draw the graph of $y = 2\sqrt{x}$.

Solution

x	0	1	2	3	4	5	6	0.25	0.5	0.75
y	0	2	2.82…	3.46…	4	4.47…	4.89…	1	1.41…	1.73…

Some extra points have been added to the table to help draw the correct shape on the steep part of the curve.



Sometimes, different graphs that you have met may be combined in a function. You will find that the shape can change a lot. This happens when the function includes reciprocals or roots as well as positive integer powers of x.

Example 28.9

Question

a Complete this table for $y = 5x - \frac{2}{x}$. Give values correct to 1 d.p. when needed.

x	–3	–2	–1	–0.5	–0.1	0.1	0.5	1	2	3
y										

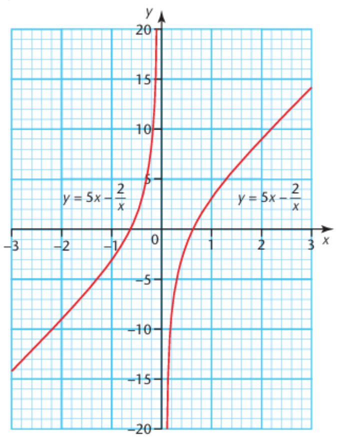
b Draw the graph of $y = 5x - \frac{2}{x}$.

Solution

a

x	−3	−2	−1	−0.5	−0.1	0.1	0.5	1	2	3
y	−14.3	−9	−3	1.5	19.5	−19.5	−1.5	3	9	14.3

b



Note

When x is near zero, the graph is very different from that of $y = 5x$. When x is far from zero, the reciprocal part of the equation has much less effect.

Exponential graphs

Exponential graphs are graphs of equations of the form

$y = ka^x$.

For the graphs that you will meet in this chapter, a is a positive integer.

Example 28.10

Question

Plot a graph of $y = 3^x$ for values of x from −2 to 3.

Use your graph to estimate

a the value of y when $x = 2.4$

b the solution to the equation $3^x = 20$.

Solution

Make a table of values and plot the graph.

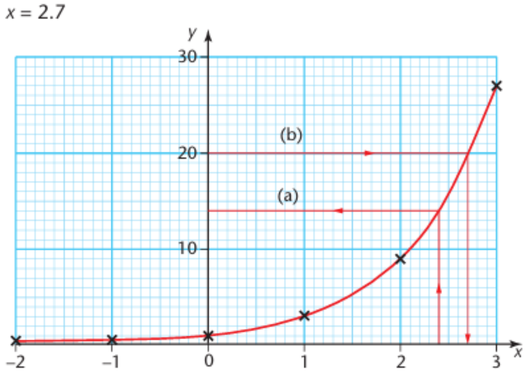
x	−2	−1	0	1	2	3
y	0.111…	0.333…	1	3	9	27

a Draw the line $x = 2.4$ and find where it intersects the curve.

$y = 14.0$

b Draw the line $y = 20$ and find where it intersects the curve.

$x = 2.7$



Growth and decay

In the function $y = a \times b^x$, the value a is repeatedly multiplied by the multiplier b.

When the multiplier is larger than 1, there is growth.

When the multiplier is less than 1, there is decay.

So exponential graphs can be used to represent and interpret growth and decay.

Example 28.11

Question

The population of a village was 700 at the beginning of 2020.
After x years, the population P is given by this equation
$P = 700 \times 1.05^x$.

a What does the number 1.05 represent in this equation?

b Draw the graph of $P = 700 \times 1.05^x$ for $x = 0$ to 10.

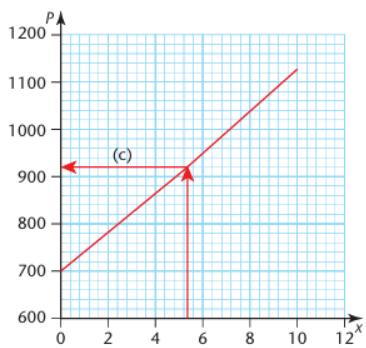
c Use your graph to estimate the population at the end of June 2025.

Solution

a 1.05 = 105%. The population is growing by 5% each year.

b

x	0	1	2	3	4	5	6	7	8	9	10
P	700	735	772	810	851	893	938	985	1034	1086	1140



Note
The P-axis has been drawn starting from 600, not 0, so that the graph can be to a larger scale. This makes values easier to read.

c Reading from the graph, when $x = 5.5$, $P = 920$.

Estimating the gradient to a curve

The gradient of a straight line is constant, and gradient $= \dfrac{\text{increase in } y}{\text{increase in } x}$.

The gradient of a curve varies. At any point, the gradient of the curve is the gradient of the tangent to the curve at that point.

So to find the gradient of a curve at a given point, first draw the tangent at the point, and then work out its gradient. Since you cannot know whether the tangent you have drawn is completely accurate, this method only gives an estimate of the gradient of the curve

Example 28.12

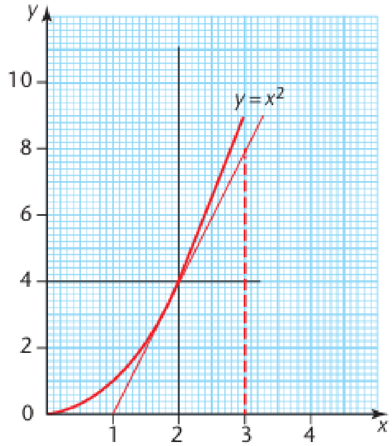
Question

Estimate the gradient of the curve $y = x^2$ at the point (2, 4).

Solution

Plot the graph and draw a tangent to the curve at (2,4).

$$\text{Gradient} = \frac{\text{increase in } y}{\text{increase in } x} = \frac{8-0}{3-1} = 4$$



Note

To draw as good a tangent as possible, try to move your ruler so you have an equal amount of space between the curve and the tangent on each side of the given point.

When choosing two points to work out the gradient of the tangent, it is easiest to make the x increase an easy number.

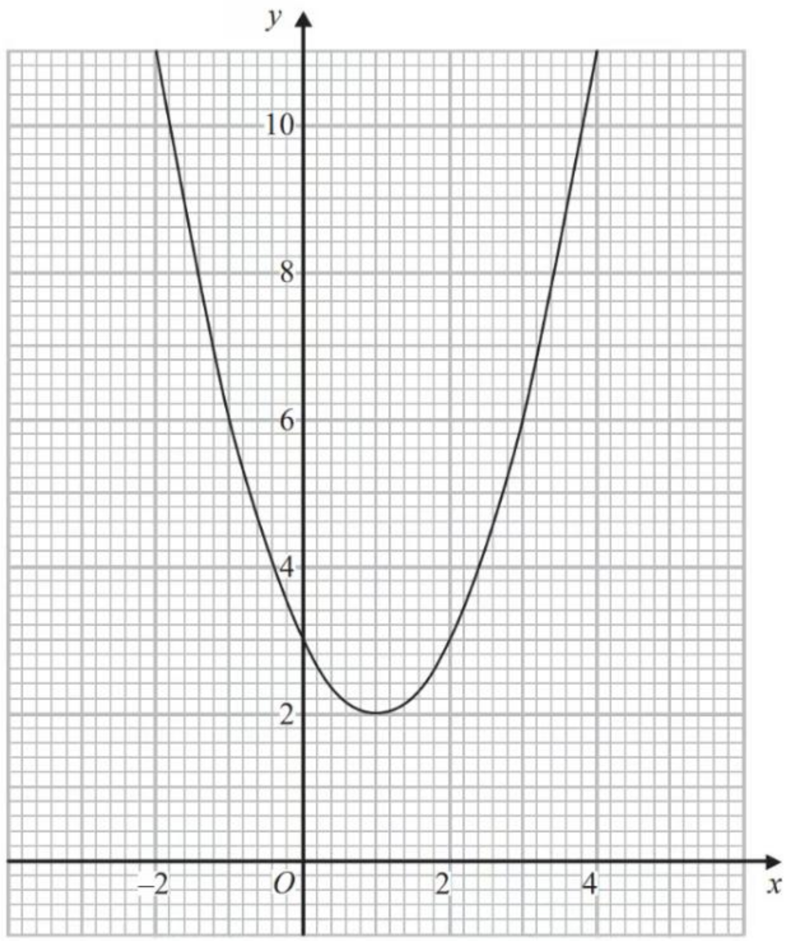
Key points

- When drawing a graph, complete a table of values to work out the points.
- Draw a smooth curve. Look ahead to the next point to help you do this.
- Graphs may be used to solve equations. Rearrange the equation if necessary so that the function you have plotted is on the left-hand side.
- Reciprocal functions have graphs with two separate branches. This is because you cannot divide any number by zero.
- Exponential curves may be used to represent growth and decay problems.
- The gradient of a curve at a given point can be estimated by drawing a tangent to the curve at that point. The gradient of the tangent is the gradient of the curve.

Revision questions

1.

The diagram shows part of the graph of $y = x^2 - 2x + 3$



By drawing a suitable straight line, use your graph to find estimates for the solutions of $x^2 - 3x - 1 = 0$

b.

P is the point on the graph of $y = x^2 - 2x + 3$ where $x = 2$

Calculate an estimate for the gradient of the graph at the point P.

2.

Complete the table of values for $y = x^2 - x - 6$

x	-3	-2	-1	0	1	2	3
y	6			-6			

b.

draw the graph of $y = x^2 - x - 6$ for values of x from -3 to 3

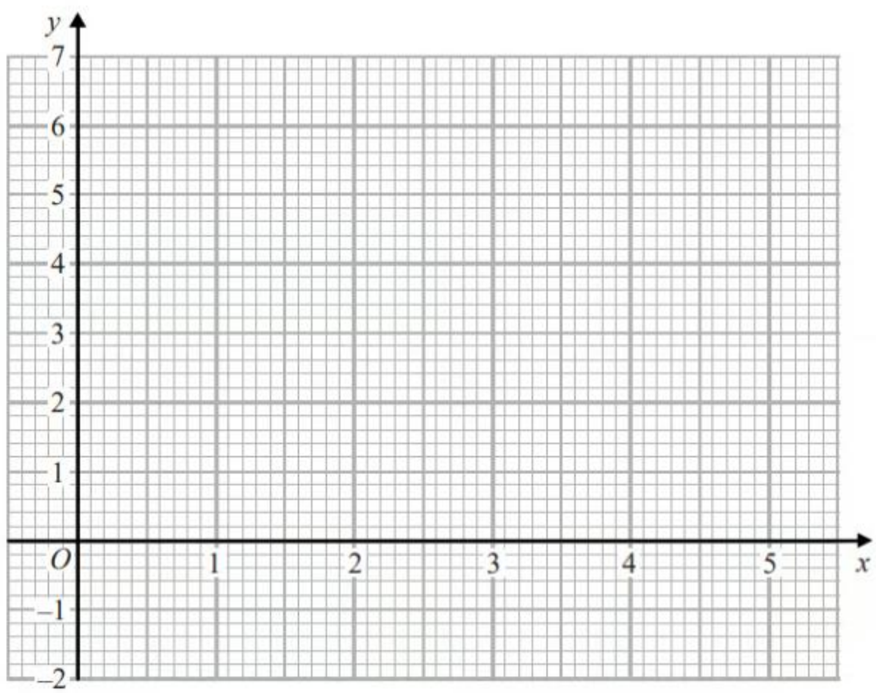
3.

Complete the table of values for $y = x^2 - 5x + 6$

x	0	1	2	3	4	5
y	6		0	0	2	

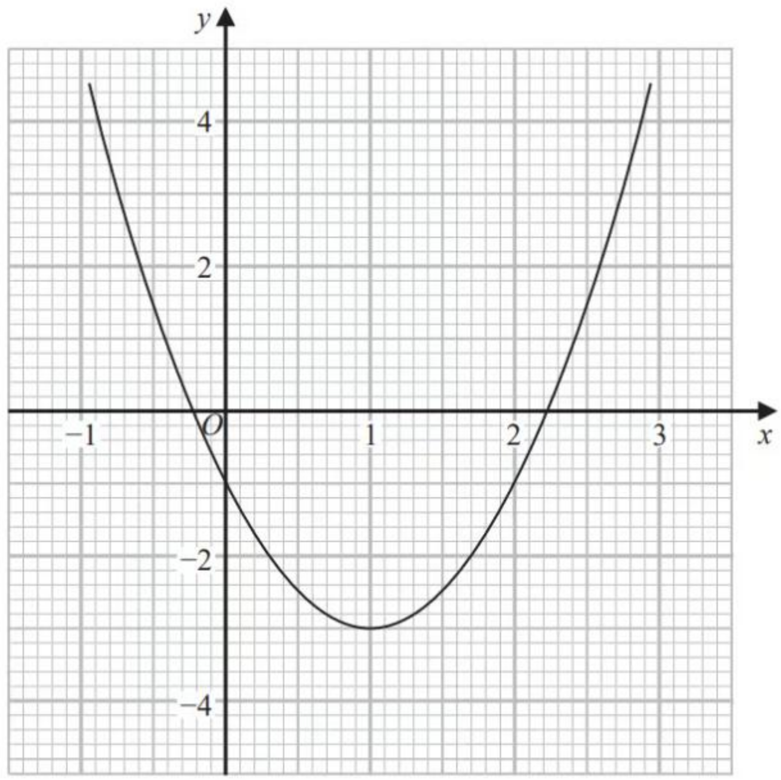
b.

On the grid, draw the graph of $y = x^2 - 5x + 6$ for $0 \leqslant x \leqslant 5$



4.

Part of the graph of $y = 2x^2 - 4x - 1$ is shown on the grid.



Use the graph to find estimates for the solutions of the equation $2x^2 - 4x - 1 = 0$

Give your solutions correct to one decimal place.

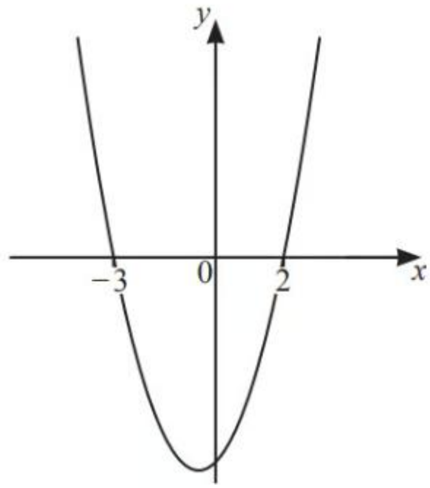
b.

By drawing a suitable straight line on the grid, find estimates for the solutions of the equation $x^2 - x - 1 = 0$

Show your working clearly.

Give your solutions correct to one decimal place.

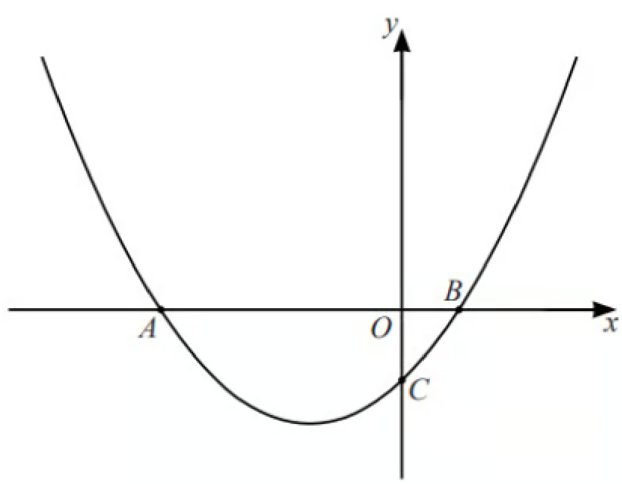
5.



This diagram shows a sketch of the graph of $y = x^2 + ax + b$.

Find the value of a and the value of b.

6.

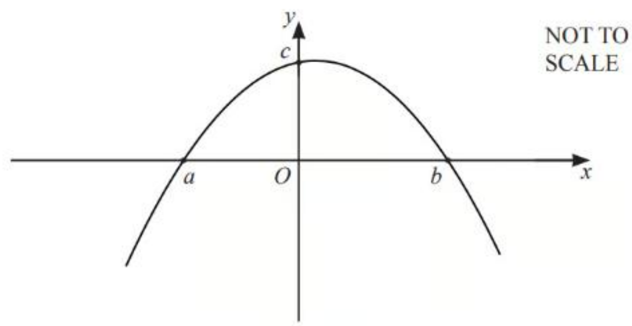


The diagram shows a sketch of the curve $y = x^2 + 3x - 4$.
Find the coordinates of the points A, B and C.

7.

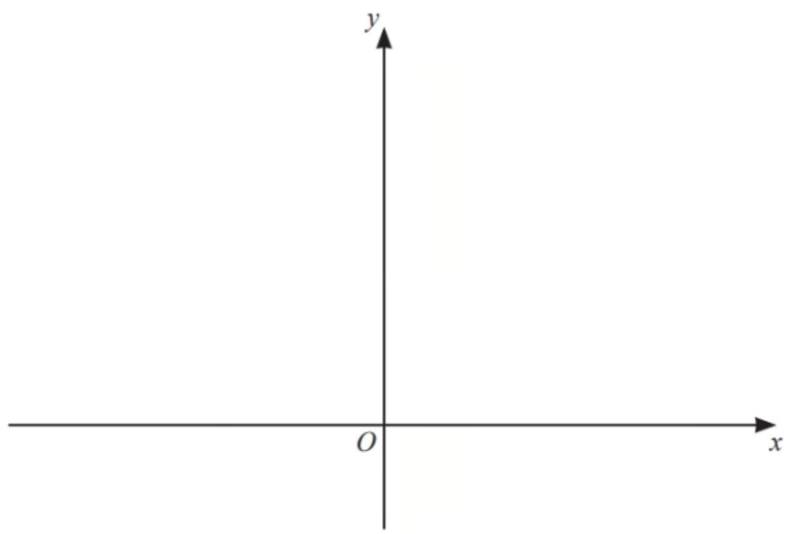
i) Factorise $24 + 5x - x^2$.

ii) The diagram shows a sketch of $y = 24 + 5x - x^2$.



Work out the values of a, b and c.

8.



On the diagram,

Sketch the graph of $y = (x - 1)^2$.

9.

The curve $y = x^2 - 2x + 1$ is drawn on a grid.
A line is drawn on the same grid.
The points of intersection of the line and the curve are used to solve the equation $x^2 - 7x + 5 = 0$.

Find the equation of the line in the form $y = mx + c$.

10.

Complete the table of values for $y = x^2 - 2x$

x	-2	-1	0	1	2	3	4
y		3	0			3	

b.

On the grid, draw the graph of $y = x^2 - 2x$ for values of x from −2 to 4