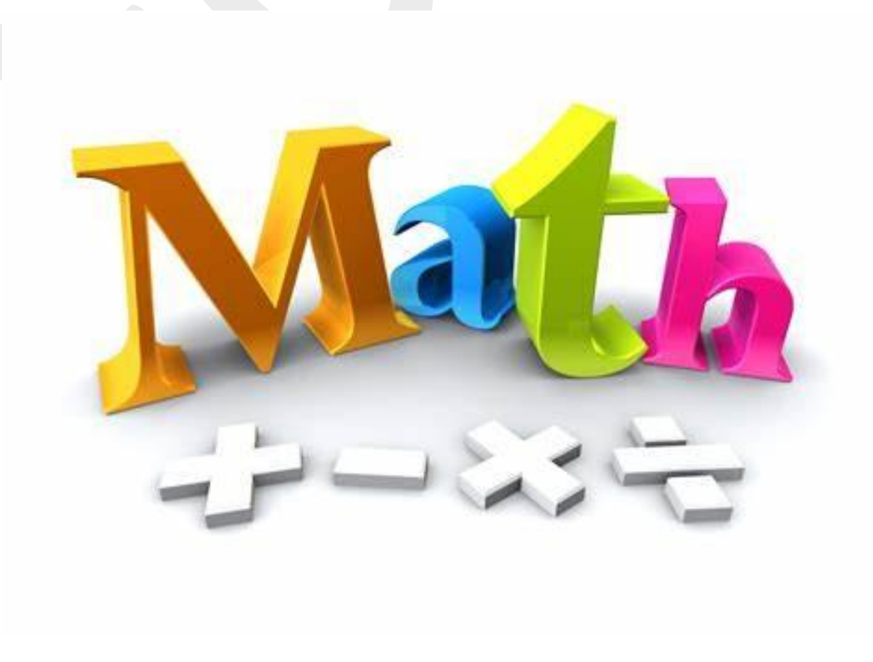
Cambridge OL

Mathematics

CODE: (4024)

Chapter 29 and Chapter 30

Sketching curves and functions



Families of graphs

You learned about these in the chapter about graphs of functions. In this chapter, you will bring together your knowledge about graphs and solving equations. By analysing a function, you can find information to help you to sketch its graph.

When you sketch a graph, you need to draw the correct shape. You also need to give other information, such as where it crosses the axes.

To find where the graph crosses the y-axis, substitute x = 0.
To find where the graph crosses the x-axis, substitute y = 0. Then, solve the resulting equation.

Linear graphs

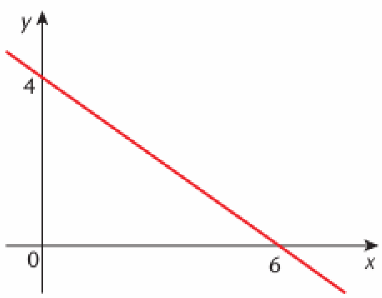
Example 29.1

Question

Sketch the graph of $2x + 3y = 12$.

Solution

When $x = 0$, $3y = 12$, so $y = 4$. The graph crosses the y-axis at (0, 4).
When $y = 0$, $2x = 12$, so $x = 6$. The graph crosses the x-axis at (6, 0).



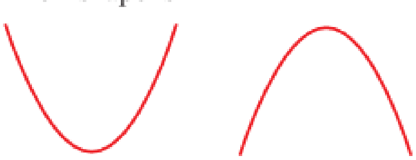
Note

When sketching lines or curves, do not scale the axes. Just label them and put on the sketch the values where the graph crosses the axes. Add the coordinates of any turning points that you find. You should rule straight lines. For curved graphs, aim for the general shape.

Quadratic graphs

These are graphs of the functions $y = ax^2 + bx + c$.

Their shape is



For $a > 0$ For $a < 0$

They are symmetrical. The line of symmetry goes through the turning point.

Example 29.2

Question

Sketch the graph of $y = 3x - x^2$. Find also the coordinates of its turning point.

Solution

The coefficient of x^2 is -1. So, the turning point is at the top of the graph.

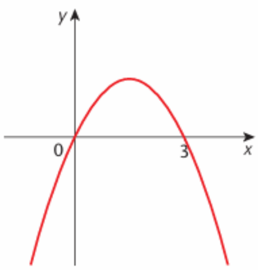
When $x = 0$, $y = 0$.

When $y = 0$, $3x - x^2 = 0$.

Factorising, $x(3 - x) = 0$.

So, $x = 0$ or 3.

Sketch:



Using symmetry, the x-value of the turning point will be halfway between 0 and 3. This is 1.5.

Substituting, $y = 3 \times 1.5 - 1.5^2 = 2.25$

The turning point is (1.5, 2.25).

Completing the square can be used to find the turning point of a quadratic graph.

Example 29.3

Question

Express $y = x^2 - 6x + 10$ in completed square form.

Hence, find the coordinates of the turning point of the graph of this function.

Then, sketch the graph.

Solution

$y = x^2 - 6x + 10$

$= (x - 3)^2 - 9 + 10$

$= (x - 3)^2 + 1$

The minimum value of a perfect square is 0.

So, the minimum value of $y = 0 + 1 = 1$.

This occurs when $x - 3 = 0$, i.e. when $x = 3$.

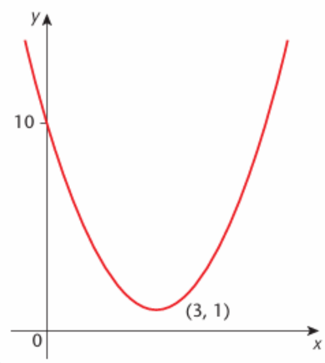
The turning point is at (3, 1) and is a minimum.

This means that the graph does not cross the x-axis.

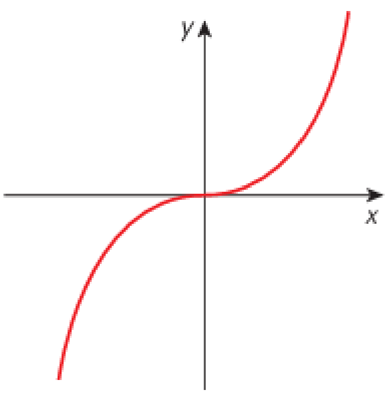
When $x = 0$, $y = 10$.

The line of symmetry goes through the turning point. Its equation is $x = 3$.

Sketch:



Cubic graphs



This is a sketch of $y = ax^3$, when a is positive.

Example 29.4

Question

Sketch the graph of $y = 8 - x^3$.

Solution

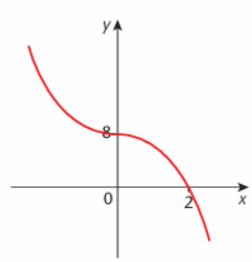
When $x = 0$, $y = 8$.

When $y = 0$, $x^3 = 8$, so $x = 2$.

When x is negative, y is positive.

When x is greater than 2, y is negative.

Sketch:



Note

Comparing this with the graph of $y = x^3$, shows that the 8 has moved the graph of $y = x^3$ up 8 units, and the negative x^3 term means that the graph of $y = x^3$ has been reflected in the y-axis.

As you know from Chapter 28, where a cubic graph crosses the axis more than once, the 'double bend' is more pronounced.

Example 29.5

Question

Find where the graph of $y = x^3 + 5x^2 - 6x$ crosses the axes.

Use your result to help you to sketch this graph.

Solution

When $x = 0$, $y = 0$.

When $y = 0$, $x^3 + 5x^2 - 6x = 0$.

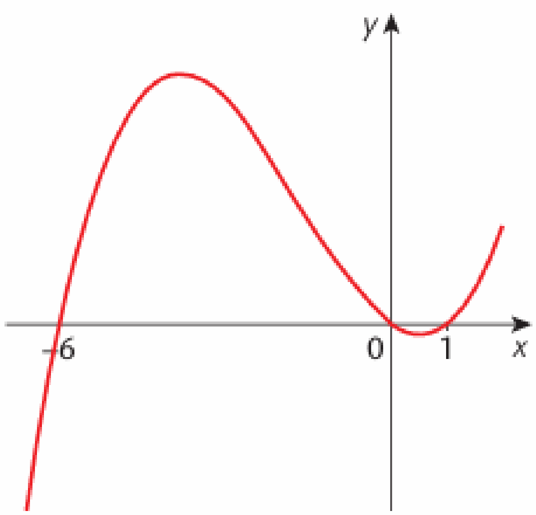
x is a factor of the left-hand side, so

$x(x^2 + 5x - 6) = 0$

$x(x + 6)(x - 1) = 0$

$x = 0, -6$ or 1.

Sketch:

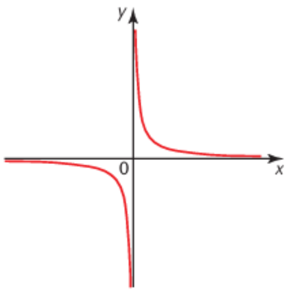


Note

Learn to find the factors of cubic expressions where x is a factor, as in this example.

Reciprocal graphs

You met the shape of these in Chapter 28.



$y = \frac{a}{x}$, where $a > 0$.

In these graphs, the curves do not meet the axes but get closer and closer to them.

Lines where this happens are called asymptotes.

These graphs are symmetrical.

For quadratic graphs, the line of symmetry is vertical.

For graphs of the form $y = \frac{a}{x}$, there are two lines of symmetry. These are the diagonal lines, $y = x$ and $y = -x$.

Example 29.6

Question

Sketch the graph of $y = \frac{-2}{x}$.

Solution

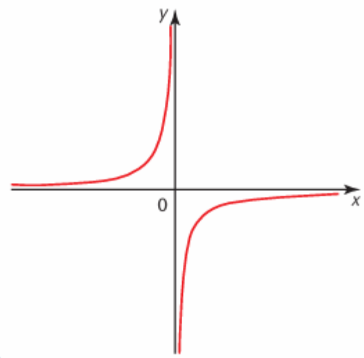
You cannot divide –2 by zero, so $x = 0$ is an asymptote.

As x gets very large, $\frac{-2}{x}$ gets close to zero, so $y = 0$ is an asymptote.

When x is negative, y is positive.

When x is positive, y is negative.

Sketch:



Example 29.7

Question

Sketch the graph of $y = \frac{12}{x} + 4$.

Solution

You cannot divide 12 by zero, so $x = 0$ is an asymptote.

When $y = 0$, $\frac{12}{x} = -4$

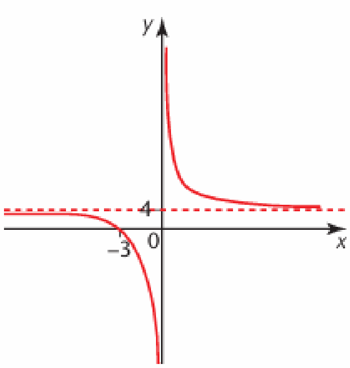
So, $12 = -4x$

$x = -3$

As x gets very large, $\frac{12}{x}$ gets close to zero, so y approaches 4.

So, $y = 4$ is the other asymptote.

Sketch:



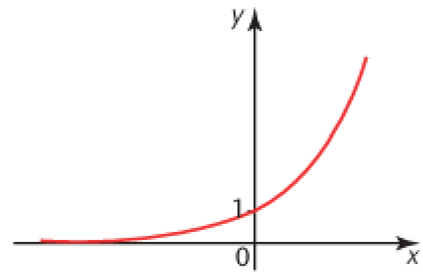
Note

Where an asymptote is not an axis, you should show it on the sketch as a dashed line.

The effect of the +4 in the equation of the graph has been to move it up by 4.

Exponential graphs

You met the shape of these in Chapter 28.



This is a sketch of $y = a^x$, where a is a positive integer.
The graph crosses the y-axis at (0, 1). The x-axis is an asymptote

Example 29.8

Question

Sketch the graph of $y = 2^x - 8$.

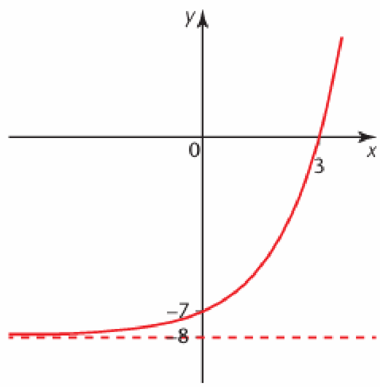
Solution

When $x = 0$, $y = 1 - 8 = -7$.

When $y = 0$, $2^x = 8$. You know that $2^3 = 8$, so $x = 3$.

For $y = 2^x$, the asymptote is $y = 0$. So, for $y = 2^x - 8$, it is $y = -8$.

Sketch:



Key points

- When you sketch a graph, you need to draw the correct shape. You also need to show where it crosses the axes.
- To find where the graph crosses the y-axis, substitute $x = 0$. To find where the graph crosses the x-axis, substitute $y = 0$. Then, solve the resulting equation.
- For quadratic graphs, you may need to use the completed square form $y = (x - a)^2 + b$. This tells you that the turning point is (a, b) and the line of symmetry is $x = a$.
- An asymptote is a line that a curve does not meet but gets closer and closer to. Reciprocal curves have two asymptotes. Exponential curves have one asymptote.
- Learn the shapes for the families of graphs in this chapter, so that you can recognise them and sketch them.

Chapter 30 - Function

Function notation

f: $x \mapsto 2x - 3$ means the **function** that maps x onto $2x - 3$.

This is often written instead as f(x) = 2x − 3. f(5) means the value of the function f when x = 5.

To find this, substitute 5 in the expression.

In the example above, this will be f(5) = 2 × 5 – 3 = 7.

Sometimes other letters, such as g and h, are used for functions.

Example 30.1

Question

You are given that f: $x \mapsto 3(x - 2)$.

a Find the value of f(6).

b Solve the equation f(x) = 6.

c Find and simplify an expression for f(x + 4).

Solution

a f(6) = 3(6 – 2)
= 3 × 4
= 12

b $3(x - 2) = 6$
$3x - 6 = 6$
$3x = 12$
$x = 4$

Note
You could start by dividing both sides by 3, giving $x - 2 = 2$.

c To find f(x + 4), substitute $x + 4$ instead of x in the expression for f.
f(x + 4) = 3(x + 4 – 2)
= 3(x + 2)

Domain and range

In the two previous functions, the starting value of x can be any real number. The set of starting values is called the domain. The set of finishing values of f(x) is called the range.

Sometimes, we want to look at the effect of using a function on a limited domain. Mapping diagrams are a useful way of representing this.

Example 30.2

Question

Draw a mapping diagram for $g(x) = 5x + 2$ for the domain {–1, 0, 1, 2, 3}.

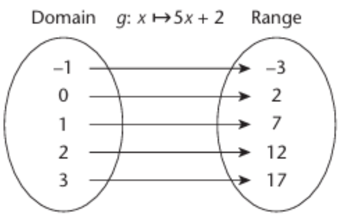
Solution

$g(-1) = 5 \times (-1) + 2 = -3$.

Similarly, $g(0) = 2$, $g(1) = 7$,
$g(2) = 12$, $g(3) = 17$.

The range is {–3, 7, 12, 17}

Mapping diagram:



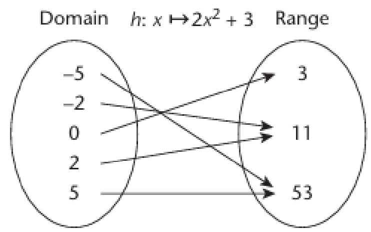
Note
Finding the values of $g(x) = 5x + 2$ for a mapping diagram is like completing a table for $y = 5x + 2$.

Example 30.3

Question

Draw a mapping diagram for the function $h(x) = 2x^2 + 3$ with domain {–5, –2, 0, 2, 5}.

Solution



Inverse functions

The inverse of a function f is the function that maps all the values of f(x) back to their original x values.

The inverse of a function is written as $f^{-1}(x)$.

For example, when $f(x) = x + 2$, $f^{-1}(x) = x - 2$.

To find the inverse function, write y in place of $f(x)$. $y = x + 2$

Rearrange the formula to make x the subject. $x = y - 2$

Replace x with $f^{-1}(x)$ and y with x to give the expression for the inverse function. $f^{-1}(x) = x - 2$

Example 30.4

Question

When $f(x) = 3x - 5$,

a find $f^{-1}(x)$ **b** find $f^{-1}(7)$.

Solution

a $y = 3x - 5$ Write y in place of $f(x)$.

$y + 5 = 3x$ Rearrange the formula to make x the subject.

$x = \frac{y+5}{3}$

$f^{-1}(x) = \frac{x+5}{3}$ Replace x with $f^{-1}(x)$ and y with x.

b $f^{-1}(7) = \frac{7+5}{3} = 4$

Note
Finding $f^{-1}(7)$ is the same as finding the value of x when $f(x) = 7$. We could solve the equation $3x - 5 = 7$ if we had not found $f^{-1}(x)$ first.

Mapping diagrams may be used to illustrate inverse functions. The range of a function is the domain of its inverse.

Example 30.5

Question

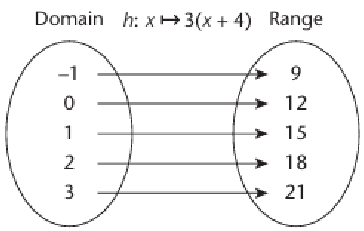
a Draw a mapping diagram for the function $h(x) = 3(x + 4)$ and domain {–1, 0, 1, 2, 3}.

b Find $h^{-1}(x)$.

c Draw the mapping diagram for this inverse function, using as the domain the range in part **a**.

Solution

a



b $h(x) = 3(x + 4)$

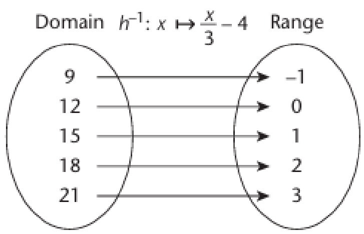
$y = 3(x + 4)$

$\frac{y}{3} = x + 4$

$x = \frac{y}{3} - 4$

$h^{-1}(x) = \frac{x}{3} - 4$

c



Note
Check that you have rearranged correctly by substituting a number from the new domain into $h^{-1}(x)$. Make sure that the result matches with its original value from $h(x)$.

In Example 30.3 you looked at the function $h(x) = 2x^2 + 3$, where both 2 and –2 in the domain had the same value in the range. The inverse function must have only one outcome for each member of its domain. So, we need to use the positive square root only when finding $h^{-1}(x)$ for functions like this one:

$h(x) = 2x^2 + 3$

$y = 2x^2 + 3$

$y - 3 = 2x^2$

$\frac{y-3}{2} = x^2$

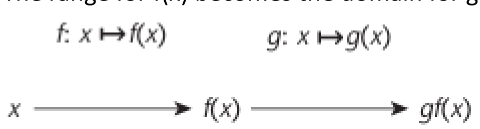
$x = \sqrt{\frac{y-3}{2}}$

$h^{-1}(x) = \sqrt{\frac{x-3}{2}}$

The domain for this inverse function is $\{x: x \geqslant 3\}$. Its range is $\{y: y \geqslant 0\}$.

Composite functions

Functions can be combined. gf(4) means first finding f(4) and then g(the result).
The range for f(x) becomes the domain for g(x) when working out gf(x).



Example 30.6

Question

$f(x) = 2x + 1$ and $g(x) = 3x - 2$

a Find gf(5).

b Find and simplify gf(x).

Solution

a $f(5) = 2 \times 5 + 1 = 11$

$gf(5) = g(11) = 3 \times 11 - 2 = 31$

b $f(x) = 2x + 1$

$gf(x) = g(2x + 1)$

$= 3(2x + 1) - 2$ use $(2x + 1)$ instead of x in the function $g(x)$

$= 6x + 3 - 2$

$= 6x + 1$

Note

You can substitute $x = 5$ into the answer in **b** to check the answer to **a**.

Key points

- $f: = x \rightarrow 3x + 1$ means the function that maps x onto $3x + 1$. It may also be written as $f(x) = 3x + 1$.
  f(2) means the value of the function when $x = 2$.
- The domain of a function f is the set of starting values of x.
  The range of a function is the set of values of $f(x)$.
- Mapping diagrams may be used to represent the effect of a function.
- The inverse of the function f is written as $f^{-1}(x)$. The inverse maps all the values of x back to their original values.
- To find the inverse function, write y in place of $f(x)$.
  Rearrange the formula to make x the subject.
  Replace x with $f^{-1}(x)$ and y with x to give the expression for the inverse function.
- Functions can be combined. gf(4) means finding first f(4) then g(the result). The range for $f(x)$ becomes the domain for $g(x)$ when working out gf(x).

Revision questions

1.

A straight line, l, has equation $y = 5x + 12$.
Write down the gradient of line l.

2.

The line $y = 3x - 2$ crosses the y-axis at G.

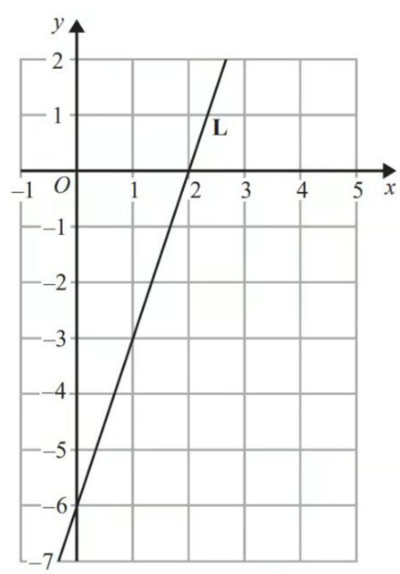
Write down the coordinates of G.

3.

Find the co-ordinates of the point where the line $y = 3x - 8$ crosses the y-axis.

4.

The line **L** is shown on the grid.



Find an equation for **L**.

5.

The equation of line P is $5x + 2y = 13$.

The equation of line Q is $y = 2x - 7$.

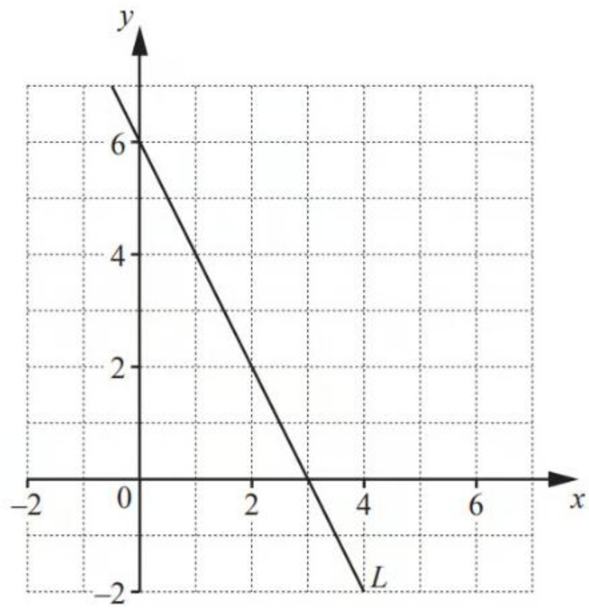
Find the gradient of line P.

6.

A is the point $(3, 5)$ and B is the point $(1, -7)$.
Find the equation of the line perpendicular to AB that passes through the point A.
Give your answer in the form $y = mx + c$.

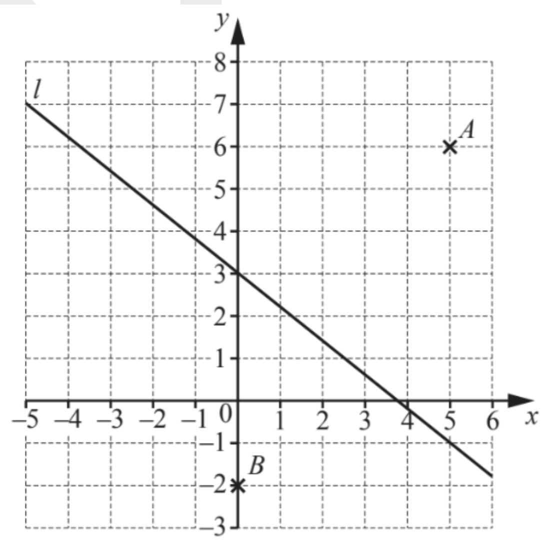
7.

The diagram shows a straight line L.



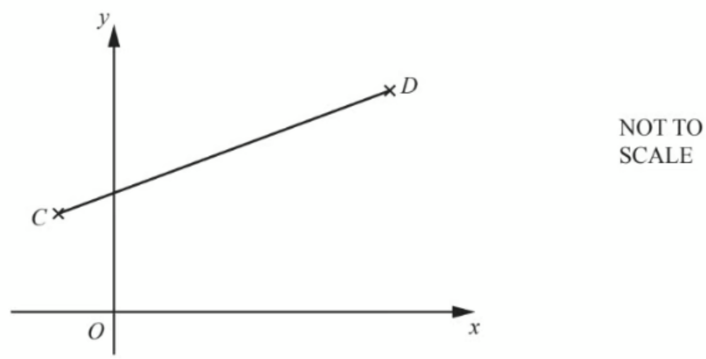
Find the equation of line L.

8.



Write down the equation of the line parallel to line l that passes through the point B.

9.



The diagram shows the points $C(-1, 2)$ and $D(9, 7)$.

Find the equation of the line perpendicular to CD that passes through the point $(1, 3)$.
Give your answer in the form $y = mx + c$.

10.

$$g(x) = 1 - 2x$$

Find the value of

Find $g(x)g(x) - gg(x)$, giving your answer in the form $ax^2 + bx + c$.

11.

$h(x) = 3^x$.

Find the value of k for which $\frac{1}{h(x)} = 9^{kx}$

12.

$f(x) = 3x + 2$ $\quad$ $g(x) = x^2 + 1$

Find $\frac{g(x)}{f(x)} + x$.
Give your answer as a single fraction, in terms of x, in its simplest form.

13.

$h(x) = x^2$

Find the values of p that satisfy $h(p) = p$.

14.

$$f(x) = 7 - 2x \qquad g(x) = \frac{10}{x}, x \neq 0 \qquad h(x) = 27^x$$

Simplify, giving your answer as a single fraction.

$$\frac{1}{f(x)} + g(x)$$

15.

$h(x) = 3^x$.

Find x when $h^{-1}(x) = -2$.

16.

$$f(x) = 7x - 2 \qquad g(x) = x^2 + 1 \qquad h(x) = 3^x$$

$gg(x) = ax^4 + bx^2 + c$

Find the values of a, b, and c.

17.

$h(x) = 3^x$.

If $h(3x) = k^x$, find the value of k.

18.

$$g(x) = 2x - 1 \qquad h(x) = 3^x$$

Find x when $h^{-1}(x) = g(2)$.

19.

The functions f and g are such that

$f(x) = 5x + 3 \quad g(x) = ax + b$ where a and b are constants.

$g(3) = 20$ and $f^{-1}(33) = g(1)$

Find the value of a and the value of b.