Cambridge OL

Mathematics

CODE: (4024)

Chapter 31 and Chapter 32

and chapter 33

- *Coordinate geometry*
- *Geometrical terms*
- *Geometrical constructions*

Chapter 31 - Coordinate geometry

The gradient of a straight-line graph

You have already found the gradient of a straight line as

$$\text{Gradient} = \frac{\text{increase in } y}{\text{increase in } x}$$

Remember that lines with a positive gradient slope up to the right and lines with a negative gradient slope down to the right.

If you know two points on the line then you do not have to draw it to work out the gradient.

For a line through two points (x_1, y_1) and (x_2, y_2), gradient $= \frac{y_2 - y_1}{x_2 - x_1}$.

Example 31.1

Question

Find the gradient of the line joining the points (3, 5) and (8, 7).

Solution

Gradient $= \frac{7-5}{8-3} = \frac{2}{5}$

Note

You can use the points the other way round in the formula. The answer will be the same.

Line segments

A line can be extended in either direction forever. It is infinite.
A line segment is the part of a line between two points.

The midpoint of a line segment

The coordinates of the midpoint of a line segment are the mean of the coordinates of the two end points.

The midpoint of the line joining the points (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Example 31.2

Question

Find the coordinates of the midpoint of the line segment with these end points.

a $A(2, 1)$ and $B(6, 7)$ **b** $C(-2, 1)$ and $D(2, 5)$

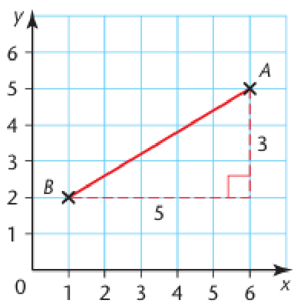
Solution

a Midpoint $= \left(\frac{2+6}{2}, \frac{1+7}{2}\right)$

$= (4, 4)$

b Midpoint $= \left(\frac{-2+2}{2}, \frac{1+5}{2}\right)$

$= (0, 3)$



The length of a line segment

You can use Pythagoras' theorem to find the length of a line segment.

The length of the horizontal side is 6 – 1 = 5.

The length of the vertical side is 5 – 2 = 3.

You can then use Pythagoras' theorem to work out the length of *AB*.

$AB^2 = 5^2 + 3^2$

$AB^2 = 25 + 9 = 34$

$AB = \sqrt{34}$

$AB = 5.83$ units to 2 decimal places.

The length of a line segment with end points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Example 31.3

Question

A is the point (–5, 4) and *B* is the point (3, 2).
Find the length *AB*.

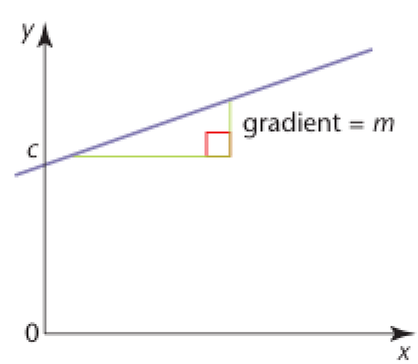
Solution

$AB = \sqrt{(-5-3)^2 + (4-2)^2} = \sqrt{(-8^2) + 2^2} = \sqrt{64+4} = \sqrt{68}$

$= 8.25$ units to 2 decimal places.

The general form of the equation of a straight line

This line has gradient m and crosses the y-axis at the point (0, c).



c is called the y-intercept because it is where the line intercepts, or crosses, the y-axis.

The equation of the line is y = mx + c. The equation of any straight line is of the form ax + by = k.

This can be rearranged to give y = mx + c.

From this, the gradient and y-intercept can be found

Example 31.4

Question

a The equation of a straight line is $y = 7$.
Find its gradient and y-intercept.

b The equation of a straight line is $5x + 2y = 10$.
Find its gradient and y-intercept.

Solution

a This can be written, $y = 0x + 7$.
So, gradient is 0 and y-intercept is 7.
[$y = 7$ is a horizontal straight line]

b $5x + 2y = 10$

$2y = -5x + 10$ Rearrange the equation into the form $y = mx + c$.

$y = -2.5x + 5$

So the gradient is -2.5 and the y-intercept is 5.

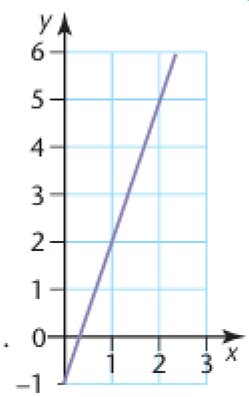
Finding the equation of a straight line

You can also use this method in reverse to find the equation of a line from its graph.

Example 31.5

Question

Find the equation of this straight line.



Solution

The gradient of a line, $m = \frac{\text{increase in } y}{\text{increase in } x}$

$= \frac{6}{2} = 3$

The line passes through (0, −1), so the y-intercept, c, is −1.

So the equation of the line is $y = 3x - 1$.

Example 31.6

Question

Find the equation of the line that passes through the points (4, 6) and (6, 2).

Solution

First find the gradient of the line.

Gradient, $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-6}{6-4} = -\frac{4}{2} = -2$

To find the y-intercept, c, substitute the coordinates of either of the given points and the gradient into the general equation of a line.

$y = mx + c$

$y = -2x + c$ The gradient is −2.

$6 = -2 \times 4 + c$ Substitute the coordinates of one of the given points.

$6 = -8 + c$

$6 + 8 = c$

$c = 14$ Solve to find c.

So $y = -2x + 14$

Or $2x + y = 14$

Parallel and perpendicular lines

Parallel lines have the same gradient. There is also a connection between the gradients of perpendicular lines.

If a line has gradient m, a line perpendicular to this has gradient $-\frac{1}{m}$.

The product of the two gradients is -1.

Note

$m \times \frac{-1}{m} = -1$

Example 31.7

Question

Find the equation of the line that is parallel to the line $2y = 3x + 4$ and passes through the point (3, 2).

Solution

The gradient of the line $2y = 3x + 4$ is 1.5.

So the equation of a line parallel to this is $y = 1.5x + c$.

$y = 1.5x + c$ — To find the y-intercept, c, substitute the coordinates of the given point into the equation.

$2 = 1.5 \times 3 + c$

$2 = 4.5 + c$

$c = 2 - 4.5$

$c = -2.5$

So, $y = 1.5x - 2.5$

Or, $2y = 3x - 5$ — Multiply both sides of the equation by 2 to eliminate the decimals.

Example 31.8

Question

Find the equation of the line that crosses the line $y = 2x - 5$ at right angles at the point (3, 1).

Solution

The gradient of the line $y = 2x - 5$ is 2.

The gradient of a line perpendicular to this is $-\frac{1}{m}$.

So the gradient of the perpendicular line is $-\frac{1}{2}$.

So the equation of the perpendicular line is $y = -\frac{1}{2}x + c$.

$y = -\frac{1}{2}x + c$ — To find the y-intercept, c, substitute the coordinates of the given point into the equation.

$1 = -\frac{1}{2} \times 3 + c$

$1 = -\frac{3}{2} + c$

$c = 1 + \frac{3}{2}$

$c = 2\frac{1}{2}$

So $y = -\frac{1}{2}x + 2\frac{1}{2}$

$x + 2y = 5$ — Multiply both sides of the equation by 2 to eliminate the fractions and rearrange to give a positive x term.

Key points

- For a line segment joining the points (x_1, y_1) and (x_2, y_2)

  Gradient $= \frac{y_1 - y_2}{x_1 - x_2}$

  Midpoint $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

  Length $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

- The general form of the equation of a straight line in a coordinate plane is $y = mx + c$, where m is the gradient and c is the intercept on the y-axis.
- When the gradient of the line is known and the y-intercept is not, it can be found by substituting the coordinates of a point on the line into the equation and rearranging.
- Parallel lines have the same gradient.
- A line perpendicular to a line with gradient m has gradient $-\frac{1}{m}$.

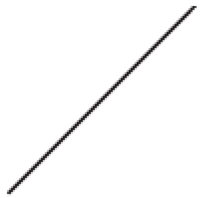
A line parallel to $y = mx + c$ will be of the form $y = mx + d$, where c and d are constants.

A line perpendicular to $y = mx + c$ will be of the form $y = -\frac{1}{m}x + e$, where c and e are constants.

Chapter 32 - 32 GEOMETRICAL TERMS

Dimensions

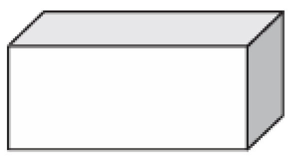
A point has no dimensions.
A line has length but no width. It has one dimension.



A **plane** has length and width but no height.

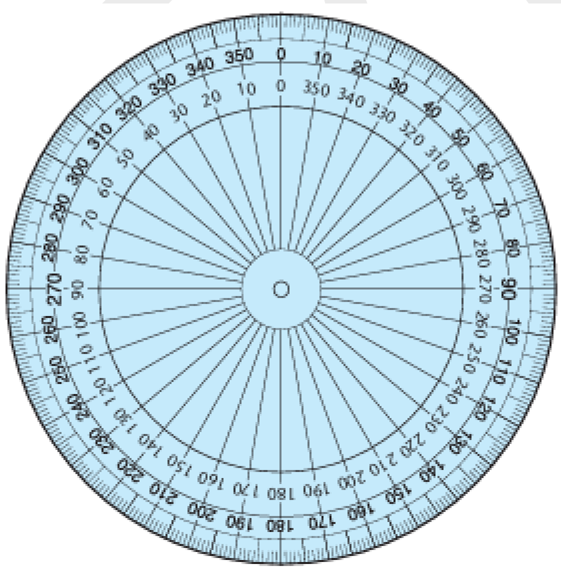
It is a flat **surface** with two dimensions.

A shape which has length, width and height has three dimensions.



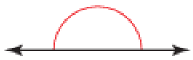
Angles

To describe and measure angles, we use a scale marked in degrees.

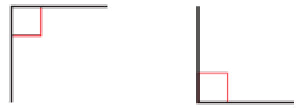


On this scale, one whole turn is equal to 360 degrees. This is written as 360°. 30 So a half turn is equal to 180° and a quarter turn is equal to 90°.

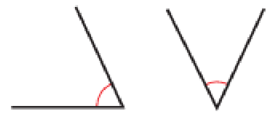
An angle of a half turn (180°) is a straight line.



Angles of a quarter turn (90°) are called **right angles**.



Angles of less than 90° are called **acute angles**.



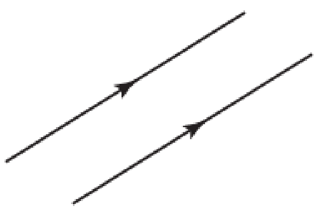
Angles of between 90° and 180° are called **obtuse angles**.

Angles of more than 180° are called **reflex angles**.

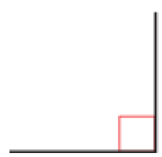
Lines

Two lines that never meet are parallel to each other.

Arrows are used to show that two lines are parallel.



Two lines that meet or cross at 90° are **perpendicular** to each other.



A line that cuts another line in half at 90° is called its **perpendicular bisector**.

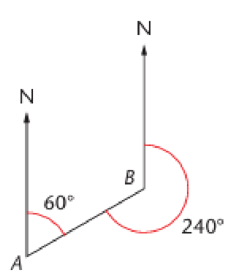
Bearings

Bearings are

- used to describe a direction
- measured in degrees from north clockwise
- made up of three digits.

Example 32.1

Question



Find the bearing of

a *B* from *A* **b** *A* from *B*.

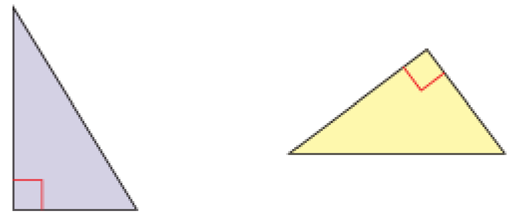
Solution

a The angle from the north line at *A* to the point *B* is 60° so the bearing of *B* from *A* is 060°.

b The angle from the north line at *B* to the point *A* is 240° so the bearing of *A* from *B* is 240°.

Triangles

A triangle is a three-sided shape.
These are right-angled triangles.

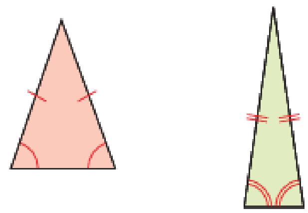


These are **isosceles triangles**.

They have two sides the same length.

They have two angles the same size.

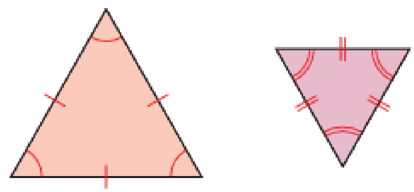
The equal sides are marked with lines.



These are **equilateral triangles**.

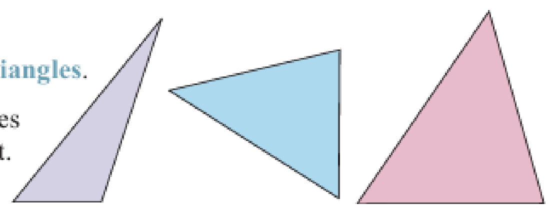
All their sides are the same length.

All their angles are the same size.



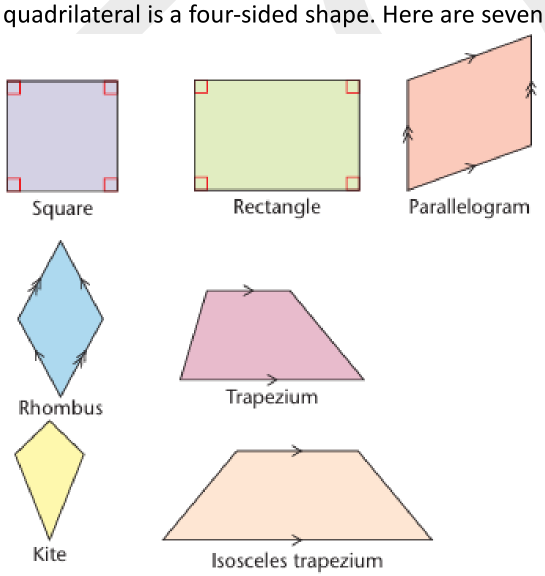
These triangles are **scalene triangles**.

In scalene triangles all the sides and all the angles are different.



Quadrilaterals

A quadrilateral is a four-sided shape. Here are seven different types.



The table shows the names and geometrical properties of the seven types of quadrilaterals shown in the diagram.

Name	Angles	Sides		Diagonals
		Lengths	Parallel	
Square	All 90°	All equal	Opposite sides parallel	Equal length Bisect at 90°
Rectangle	All 90°	Opposite sides equal	Opposite sides parallel	Equal length Bisect but not at 90°
Parallelogram	Opposite angles equal	Opposite sides equal	Opposite sides parallel	Not equal Bisect but not at 90°
Rhombus	Opposite angles equal	All equal	Opposite sides parallel	Not equal Bisect at 90°
Trapezium	Can be different	Can be different	One pair of sides parallel	Nothing special
Kite	One pair of opposite angles equal	Two pairs of adjacent sides equal	None parallel	Not equal Only one bisected by the other They cross at 90°
Isosceles trapezium	Two pairs of adjacent angles equal	One pair of opposite sides equal	Other pair of opposite sides parallel	Equal length Do not bisect or cross at 90°

Example 32.2

Question

Which quadrilaterals have

a both pairs of opposite sides equal

b just one pair of opposite angles equal?

Solution

a Square, rectangle, parallelogram, rhombus

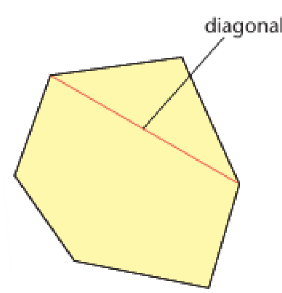
b Kite

Polygons

A polygon is a many-sided shape.

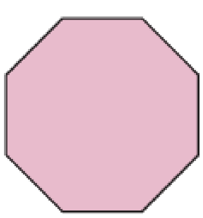
Here are some common ones, apart from the triangles and quadrilaterals you have met already.

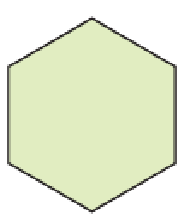
Shape	Pentagon	Hexagon	Octagon	Decagon
Number of sides	5	6	8	10

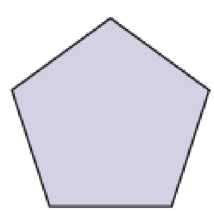


When the sides of a polygon are all the same length and the angles of the polygon are all the same, it is called a regular polygon. Otherwise, the polygon is an irregular polygon.

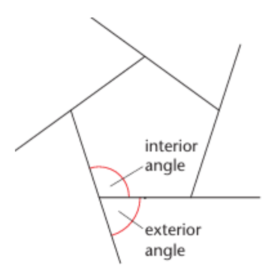
A line joining two corners of a polygon is called a diagonal.





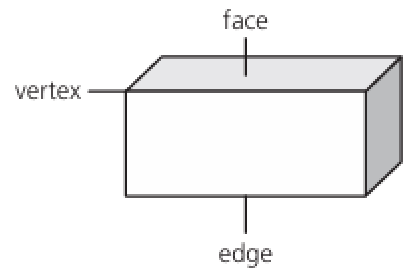


The angles inside a polygon are called interior angles. If a side of the shape is continued outside the shape, the angle made is called an exterior angle.



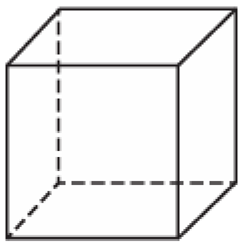
Solids

Here is a cuboid



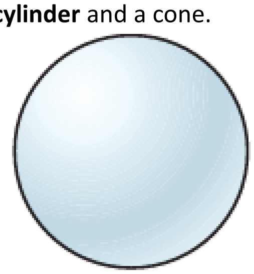
On this diagram of the cuboid, there are three hidden edges, three hidden faces and one hidden vertex. On this diagram of a cube, the hidden edges have been shown with dashed lines.

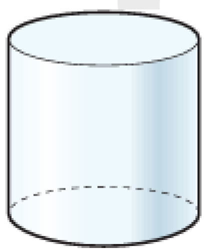
All the faces of a cube are squares

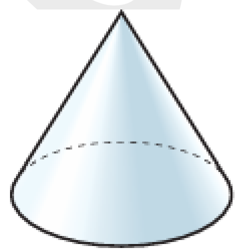


The cube and cuboid are examples of three-dimensional solids where all the faces are planes. Some solids have curved surfaces.

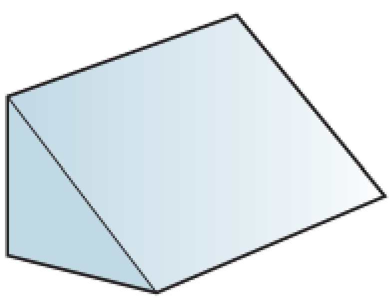
A **sphere** has only a curved surface. A **hemisphere** (half a sphere) has both a flat and a curved surface, as do a **cylinder** and a cone.

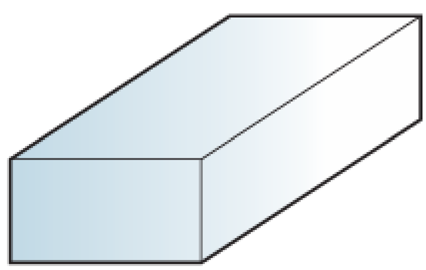

Sphere


Cylinder


Cone

Prisms and pyramids are also three-dimensional solid shapes. Here are two examples of prisms.


Triangular prism


Rectangular prism

Look at the shapes of the faces.
The shape on each end of the triangular prism is a triangle. All the other faces are rectangles.

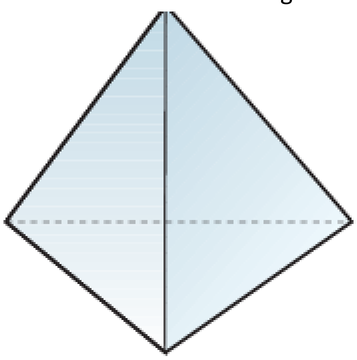
If you cut through the triangular prism at right angles to one of the rectangular faces you will see its cross-section. It is the same shape as the triangle at either end of the prism.

The cross-section of a prism is the same all the way through.

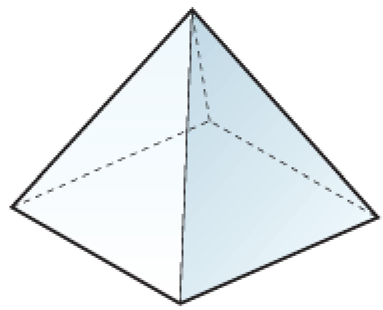
The cross-section of a prism can be any shape. The cross-section of the rectangular prism shown above is a rectangle.

A rectangular prism is called a cuboid.

A pyramid has a base and triangular faces. The base can have any number of sides.

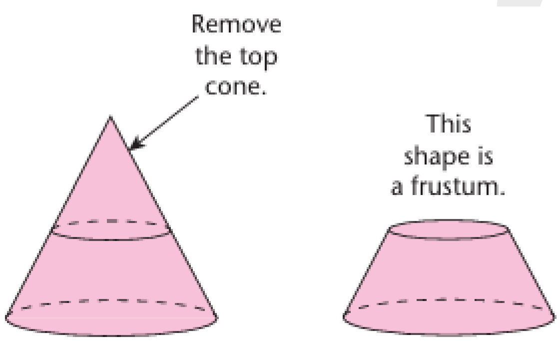


Triangle-based pyramid



Square-based pyramid

A **frustum** is a cut-off shape, cut parallel to the base of a cone or pyramid. The frustum of a cone has two flat surfaces and a curved surface. The surfaces of a frustum of a pyramid are all flat.



Nets of 3-D shapes

Three-dimensional objects with flat sides can be made by folding up a suitable flat object. This is called a net.

Example 32.3

Question

Which of these can be made from a net?

a a cuboid

b a sphere

c a square-based pyramid

d a cylinder.

Solution

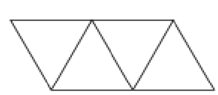
a and **c**. The others have curved surfaces.

Example 32.4

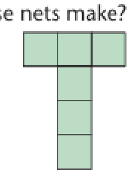
Question

Which 3-D shape does each of these nets make?

a

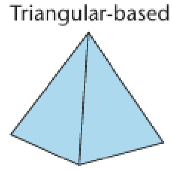


b

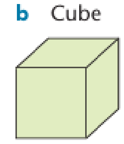


Solution

a Triangular-based pyramid



b Cube



Congruence and similarity

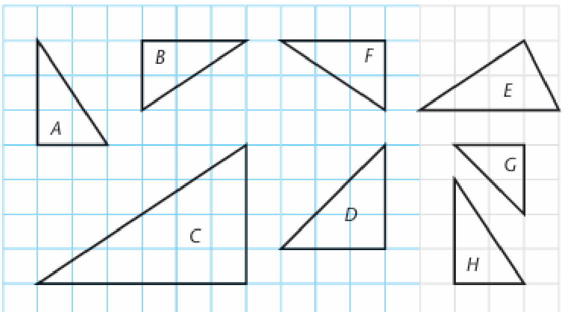
Shapes that are exactly the same size and shape are said to be congruent. If you cut out two congruent shapes, one shape would fit exactly on top of the other.

The corresponding sides are equal and the corresponding angles are equal.

Example 32.5

Question

Which of these triangles are congruent to triangle *A*?

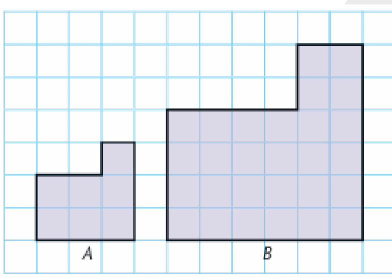


Solution

B, *F* and *H*.

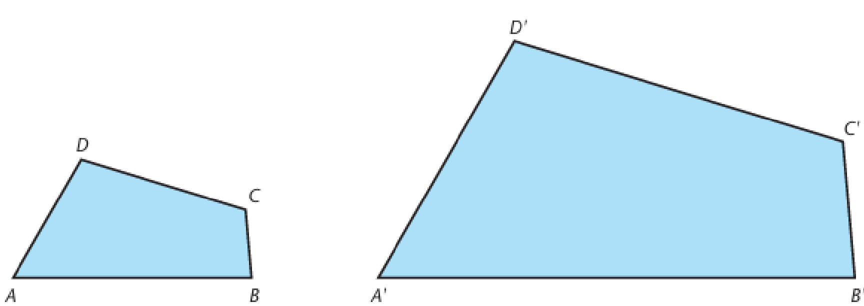
They are exactly the same with corresponding sides equal, but are a different way round.

Every length in shape B is two times as long as the corresponding length in shape A.



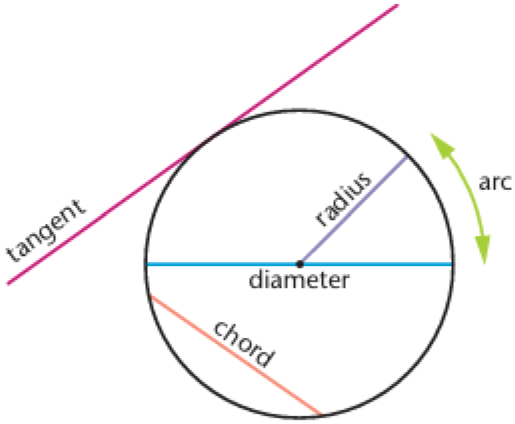
Shape B is an enlargement of shape A with scale factor 2. We can also say that shapes A and B are similar.

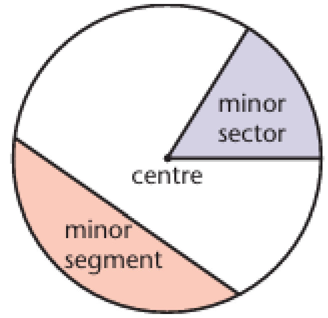
Shapes like these two quadrilaterals, which have the same angles and side lengths that are in proportion, are similar. One is an enlargement of the other.



Circles

You need to be able to identify the parts of a circle.





The **circumference** of a circle is the distance all the way round – the perimeter of the circle.

A chord divides a circle into two segments. The larger one is called the major segment and the smaller one is the minor segment.

Two radii divide a circle into two sectors. The larger one is called the major sector and the smaller one is the minor sector. The radii also divide the circumference into a major arc and a minor arc.

Key points

- A point has no dimensions. A line has one dimension (1-D). A flat shape, or plane, has two dimensions (2-D). A shape with length, width and height has three dimensions (3-D).
- Angles are measured in degrees. A full turn is 360°. A half-turn (which forms a straight line) is 180°. A quarter-turn is 90°, which is also called a right angle.
- Acute angles are less than 90°. Obtuse angles are between 90° and 180°. Reflex angles are more than 180°.
- Parallel lines never meet. Perpendicular lines meet or cross at 90°. A line that cuts another line in half at 90° is called its perpendicular bisector.
- Bearings are used to describe a direction. They are measured clockwise round from North and are always given with three digits.
- There are different types of triangle: right-angled, isosceles, equilateral and scalene.
- There are different types of quadrilateral: square, rectangle, parallelogram, rhombus, trapezium, kite, isosceles trapezium, as well as scalene.
- Pentagons, hexagons, octagons and decagons are all polygons.
- Regular polygons have all their sides the same length and all their angles the same size. Otherwise, a polygon is irregular.
- Angles inside a polygon are called interior angles. Exterior angles are made when a side of a shape is continued outside the shape.
- Congruent shapes are exactly the same shape and size.
- Similar shapes have the same angles but their sizes are different. The sides are in proportion to each other. So, when two shapes are similar, one is an enlargement of the other.
- 3-D solids that you should know about include: cube, cuboid, sphere, hemisphere, cylinder, cone, prism, pyramid and frustum.
- A 'corner' in a shape is called a vertex (the plural is 'vertices'). For 2-D shapes, a vertex is where two sides meet. For 3-D shapes, a vertex is where three or more edges meet.
- A flat surface on a 3-D shape is called a face.
- A net of a 3-D shape is a 2-D shape that can be folded to make the 3-D shape.
- Circle terms that you should know about include: centre, radius (plural radii), diameter, circumference, semi-circle, chord, tangent; also, major and minor arcs, sectors and segments.

Chapter 33 - GEOMETRICAL CONSTRUCTIONS

Measuring angles

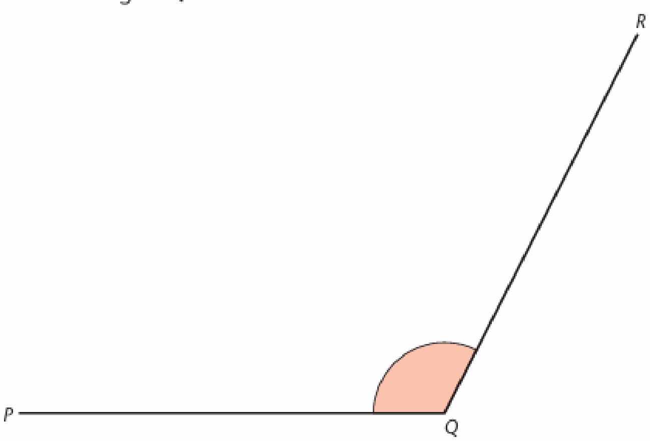
The instrument used to measure an angle is called a protractor or an angle measurer. Some protractors are full circles and can be used to measure angles up to 360°.

Most protractors are semi-circular in shape and can be used to measure angles up to 180°.

Example 33.1

Question

Measure angle *PQR*.



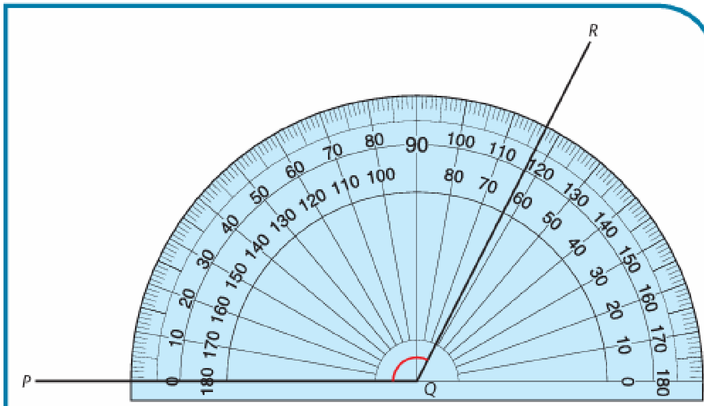
Solution

First make an estimate.

The angle is obtuse and so will be between 90° and 180°.

A rough estimate is about 120°.

Place your protractor so that the zero line is along one of the arms of the angle and the centre is at the point of the angle.



Start at zero.

Go round this scale until you reach the other arm of the angle.

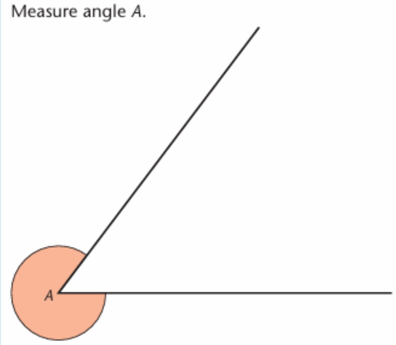
Then read the size of the angle from the scale.

Angle *PQR* = 117°

Example 33.2

Question

Measure angle *A*.



Solution

A reflex angle is between 180° and 360°.

This reflex angle is over $\frac{3}{4}$ of a turn so it is bigger than 270°.

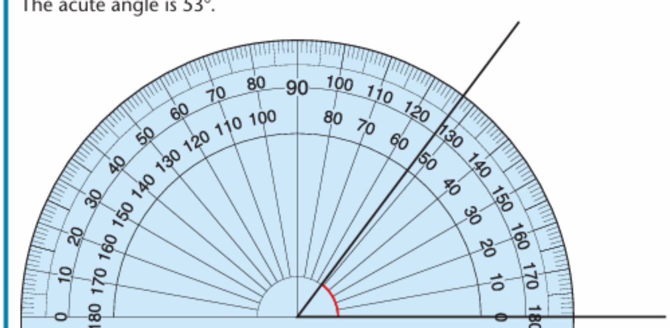
A rough estimate is 300°.

You can measure an angle of this size directly using a 360° circular angle measurer. However, the scale on a semi-circular protractor only goes up to 180°.

You need to do a calculation as well as measure an angle.

Measure the acute angle first.

The acute angle is 53°.



The acute angle and the reflex angle together make 360°.

Use the fact that the two angles add up to 360° to calculate the reflex angle.

Angle *A* = 360° – 53°

= 307°

Constructing a geometrical figure using compasses

Constructing a triangle given three sides

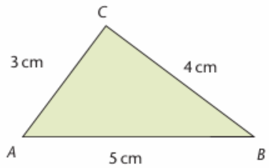
Example 33.3

Question

Construct triangle *ABC*, where *AB* = 5 cm, *BC* = 4 cm and *AC* = 3 cm.

Solution

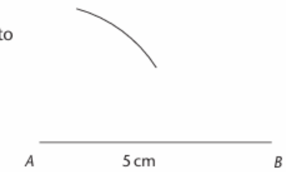
First draw a sketch.

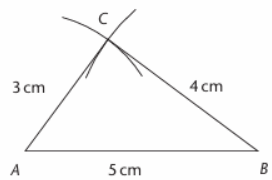


Draw the line *AB*, 5 cm long. From *A*, with your compasses set to a radius of 3 cm, draw an arc above the line.

From *B*, with your compasses set to a radius of 4 cm, draw another arc to intersect the first. The point where the arcs meet is *C*.

Join points *A* and *B* to point *C* to complete the triangle.





Note

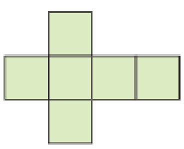
Use a ruler for all straight lines. Leave in all your construction arcs.

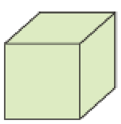
Note

A common error in measuring angles is to use the wrong scale of the two on the protractor. Make sure you choose the one that starts at zero. A useful further check is to estimate the angle first. Knowing approximately what the angle is should prevent you using the wrong scale.

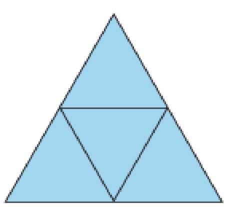
Nets of 3-D shapes

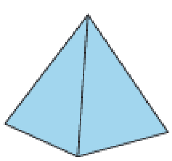
A net of a three-dimensional object is a flat shape that will fold to make that object. This flat shape can be folded to make a cube – it is a net of the cube





A regular triangular-based pyramid has edges that are all the same length. Here is its net.





As a cuboid has six faces, its net needs six rectangles. But not every arrangement of the six rectangles will fold together to make the cuboid.

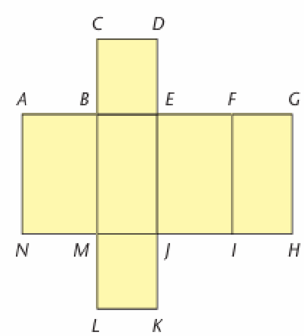
Example 33.4

Question

This net can be folded to make a cuboid.

When it is folded, which point or points will meet with

a point *A* **b** point *D*?



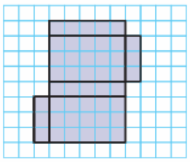
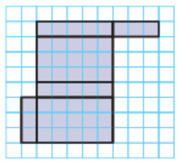
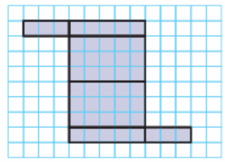
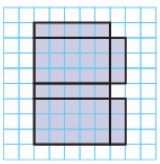
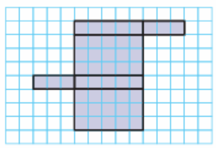
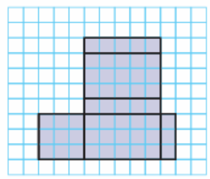
Solution

a Points *C* and *G* **b** Point *F*

Example 33.5

Question

For each of these arrangements, say whether it is the net of a cuboid.
They are drawn on squared paper to size.

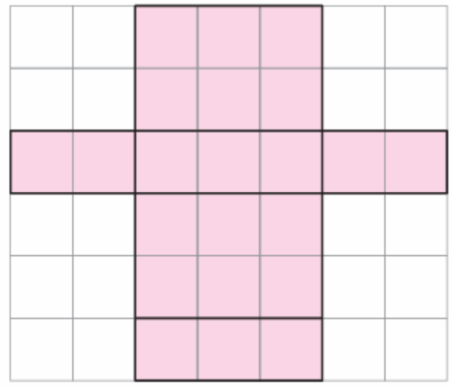
a

b

c

d

e

f


Solution

a Yes **b** Yes **c** No
d No **e** Yes **f** No

Example 33.6

Question



This net of a cuboid is drawn on a one-centimetre square grid.

a Write down the dimensions of the cuboid.
b By finding the area of the net, find the surface area of the cuboid.
c Find the volume of the cuboid.

Solution

a 3 cm by 2 cm by 1 cm
b Surface area = $2 \times (3 \times 2) + 2 \times (3 \times 1) + 2 \times (2 \times 1) = 22\,\text{cm}^2$ [or by counting squares]
c Volume = $3 \times 2 \times 1 = 6\,\text{cm}^3$

Key points

- Straight lines should be drawn and measured to an accuracy of 1 mm.
- Angles between lines should be drawn and measured to an accuracy of 1°.
- You can draw a triangle when you know the lengths of three sides. You need to use a pair of compasses to find the position of the third vertex.
- For all geometrical constructions, use a ruler for all straight lines and leave in any construction arcs.
- A net is a flat shape that will fold up to make a three-dimensional (3-D) object.

Note

If you are not sure whether a net is correct, cut it out and try folding it.

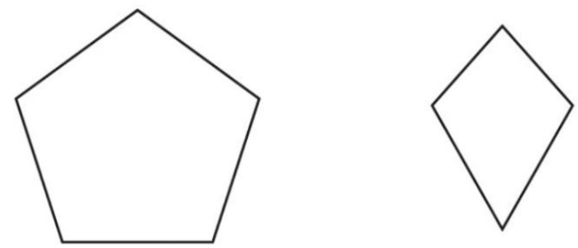
Revision questions

1. Complete each statement.

a. A quadrilateral with only one pair of parallel sides is called a

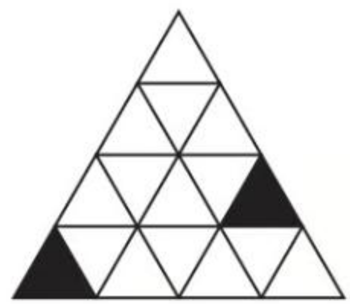
b. An angle greater than 90° but less than 180° is called.

2.
a. The diagram shows a regular pentagon and a kite. Complete the following statements.
The regular pentagon has.

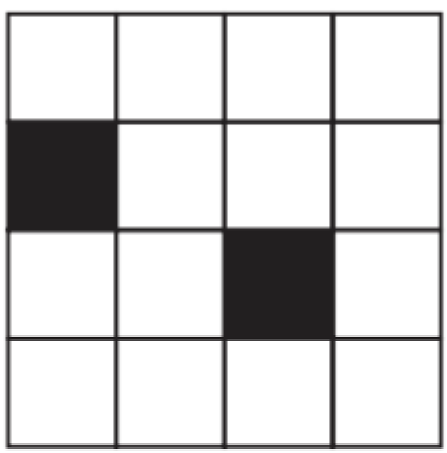


b. The kite has rotational symmetry of order
lines of symmetry.

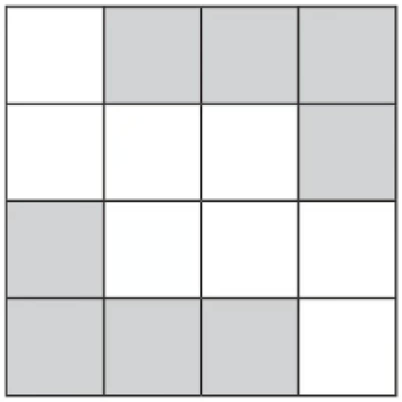
3. In the diagram, two small triangles are shaded.
Shade one more small triangle, so that the diagram will then have one line of symmetry.



4. In the diagram, two small squares are shaded.
Shade two more small squares, so that the diagram will then have rotational symmetry of order 2.

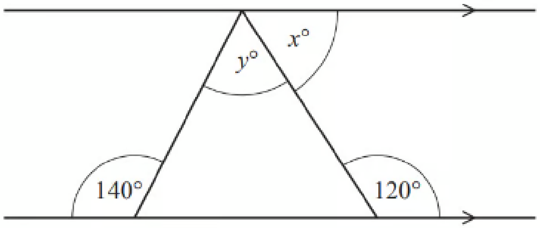


5.



Write down the order of rotational symmetry of the diagram.

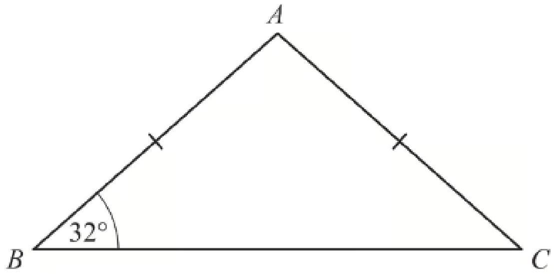
6.



NOT TO SCALE

The diagram shows a triangle drawn between a pair of parallel lines.
Find the value of x and the value of y.

7.

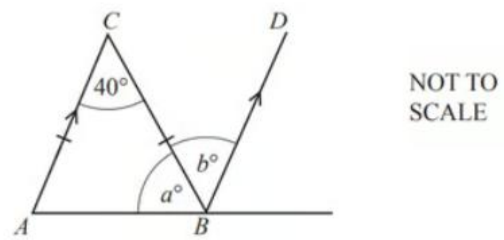


NOT TO SCALE

Triangle ABC is isosceles.
Angle $ABC = 32°$ and $AB = AC$.

Find angle BAC.

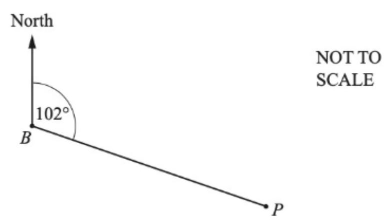
8.



Triangle ABC is isosceles.
AC is parallel to BD.

Find the value of a and the value of b.

9.



The bearing of P from B is 102°.

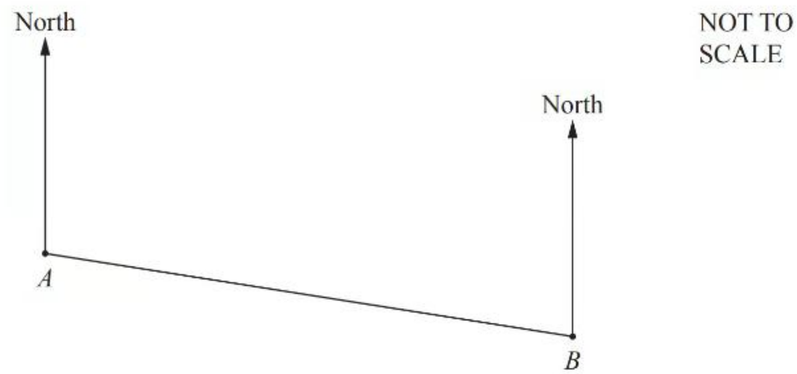
Find the bearing of B from P.

10.

A field, ABC, is in the shape of a triangle.
$AC = 500\text{m}$ and $BC = 650\text{m}$.

Using a ruler and compasses only, complete the scale drawing of the field ABC.
Leave in your construction arcs.
Use a scale of 1 cm to represent 100m.
The side AB has been drawn for you.

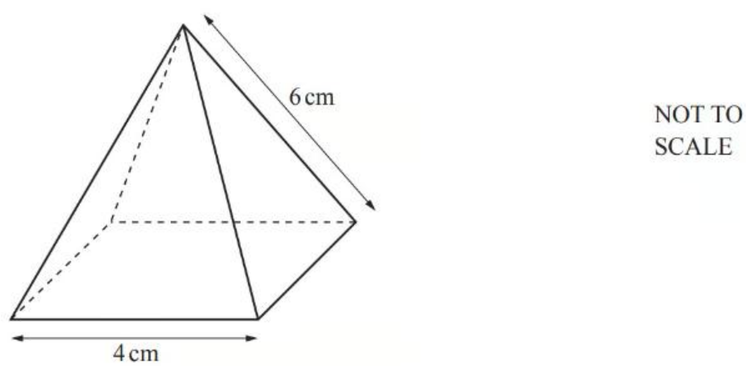
11.



The bearing of *B* from *A* is 105°.

Find the bearing of *A* from *B*.

12.



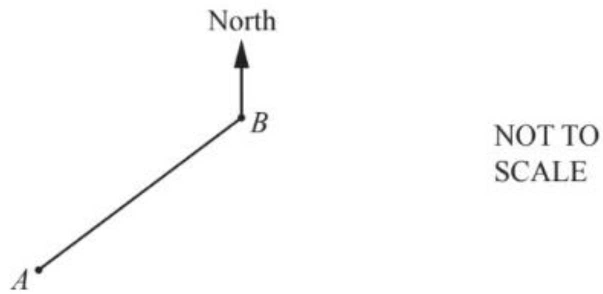
The diagram shows a pyramid with a square base.
The triangular faces are congruent isosceles triangles.
Using a ruler and compasses only, construct an accurate drawing of one of the triangular faces of the pyramid.

13.

The bearing of Alexandria from Paris is 128°.

Calculate the bearing of Paris from Alexandria.

14.

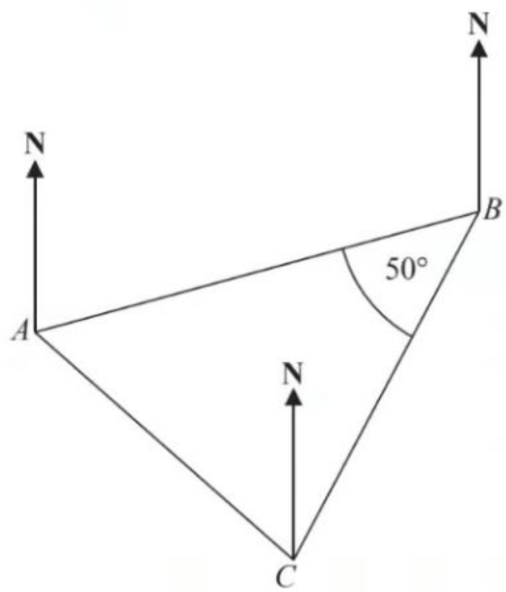


The bearing of A from B is 227°.

Find the bearing of B from A.

15.

The diagram shows the positions of three points, A, B and C, on a map.



The bearing of B from A is 070°.

Angle ABC is 50°.

$AB = CB$

Work out the bearing of C from A.