Cambridge OL

Mathematics

CODE: (4024)

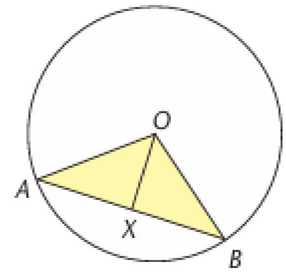
Chapter 38 and Chapter 39

Circle theorems and Units of measure

Chapter 38 - CIRCLE THEOREMS

Symmetry properties of circles

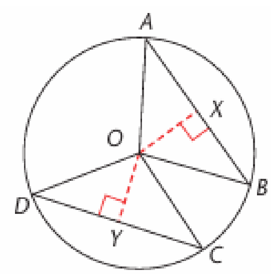
The perpendicular from the centre to a chord



AB is a chord to the circle, centre O.
X is the midpoint of AB.
OA and OB are radii, so triangle OAB is isosceles.
OX is a line of symmetry for triangle OAB.
Therefore, OX is perpendicular to AB

the perpendicular from the centre to a chord bisects the chord.

Equal chords



AB and CD are chords to the circle, centre O.

Chords AB and CD are of equal length.

X is the midpoint of AB and Y is the midpoint of CD.

OA, OB, OC and OD are radii so are equal, so triangles OAB and OCD are congruent (SSS).

Therefore $OX = OY$.

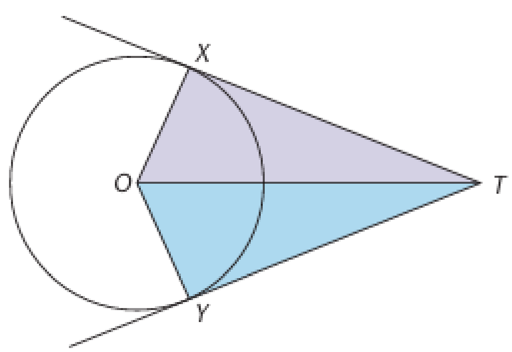
This shows that

equal chords are equidistant from the centre of the circle.

Tangents from an external point

T is a point outside a circle, centre O.
TX and TY are the tangents from T to the circle.

OX and OY are radii so are equal.



OT is a line of symmetry and bisects $X\hat{O}Y$, so $X\hat{O}T = Y\hat{O}T$.

OT is common to triangles OXT and OYT.

So triangles OXT and OYT are congruent (SAS).

Therefore $XT = YT$.

This shows that

tangents to a circle from an external point are equal in length.

These facts can be used to solve geometrical problems.

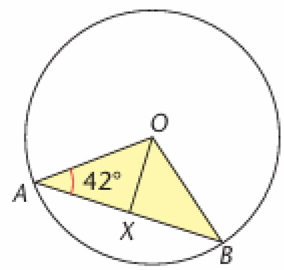
Example 38.1

Question

X is the midpoint of chord AB.

Given that $O\hat{A}B = 42°$, find

a $O\hat{B}A$ **b** $A\hat{O}B$ **c** $A\hat{O}X$.



Solution

a $O\hat{B}A = 42°$ Triangles AOX and BOX are congruent.

b $A\hat{O}B = 96°$ Angles in triangle AOB add up to 180°.

c $A\hat{O}X = 48°$ By symmetry, $A\hat{O}X = B\hat{O}X$.

Angles in a circle

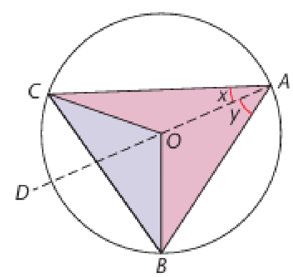
The angle subtended by an arc at the centre is twice the angle subtended at the circumference BC is a chord of the circle whose centre is O.

Arc BC subtends angle BOC at the centre of the circle and angle BAC at the circumference and AD is a straight line. Here is a simple proof that angle COB = twice angle CAB.

Let angle CAO = x and angle BAO = y.

Let angle $CAO = x$ and angle $BAO = y$.

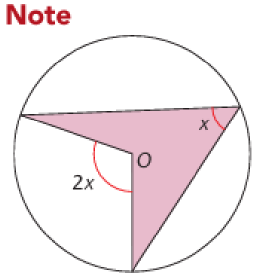
Then angle $CAB = x + y$.



Triangles OAB and OAC are isosceles.	OA, OB and OC are radii of the circle.
Angle $ACO = x$ and angle $ABO = y$	Base angles of isosceles triangles.
Angle DOC = angle OAC + angle $OCA = 2x$ Angle DOB = angle OAB + angle $OBA = 2y$	The exterior angle of a triangle equals the sum of the opposite interior angles.

$$\begin{aligned}\text{Angle } COB &= \text{angle } DOC + \text{angle } DOB \\ &= 2x + 2y \\ &= 2(x + y) \\ &= 2 \times \text{angle } CAB\end{aligned}$$

Note



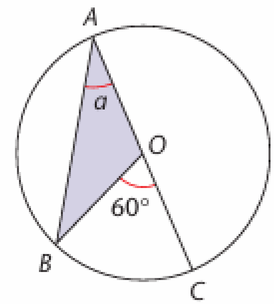
Note

An angle is formed or **subtended** at a particular point by an arc when straight lines from its ends are joined at that point.

Example 38.2

Question

In the diagram, O is the centre of the circle.
Calculate the size of angle a.

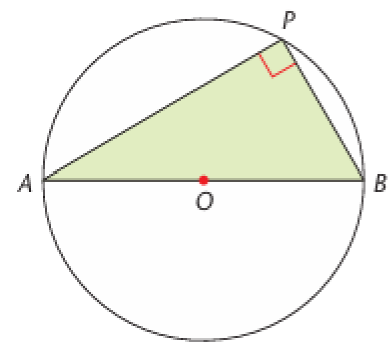


Solution

Angle $a = 30°$ Angle at the centre = twice the angle at the circumference.

The angle subtended at the circumference in a semi-circle is a right angle

This is a special case of the angle subtended by an arc at the centre is twice the angle subtended at the circumference.



Angle $AOB = 2 \times$ angle APB	The angle at the centre is twice the angle at the circumference.
Angle $AOB = 180°$	AB is a diameter, that is, a straight line.
Angle $APB = 90°$	Half of angle AOB.

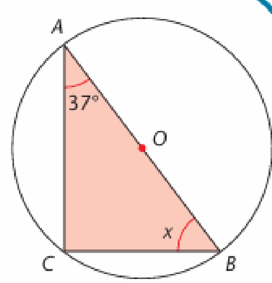
So the angle in a semi-circle is 90°.

This theorem is often used in conjunction with the angle sum of a triangle, as is shown in the next example.

Example 38.3

Question

Work out the size of angle x.
Give a reason for each step of your work.



Solution

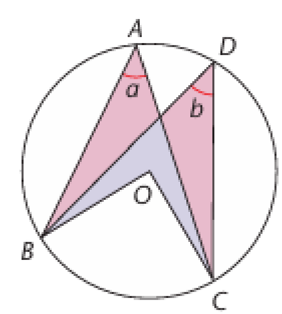
Angle $ACB = 90°$	The angle in a semi-circle is a right angle.
$x = 180 - (90 + 37)$	The angle sum of a triangle is 180°.
$x = 53°$	

Note

When calculating an angle, never assume something is true, for example, 'because it looks like it' on the diagram. You must always give a reason for each step of your calculation.

Angles in the same segment of a circle are equal

In the diagram, a = b, as both are equal to half the angle subtended at the centre. To use this property, you have to identify angles subtended by the same arc since these are in the same segment. This is shown in the next example.

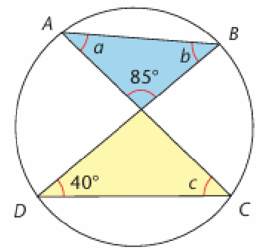


Example 38.4

Question

Find the sizes of angles a, b and c.

Give a reason for each step of your work.



Solution

a = angle $BDC = 40°$ Angles in the same segment are equal.

$b = 180 - (40 + 85)$

$= 55°$ The angle sum of a triangle is 180°.

$c = b = 55°$ Angles in the same segment are equal.

Opposite angles of a cyclic quadrilateral are supplementary

A cyclic quadrilateral has all four vertices on a circle.

This property is also derived from the fact that the angle at the centre is twice the angle at the circumference

In the diagram

angle $ABC = \frac{1}{2}x$ The angle at the centre is twice the angle at the circumference.

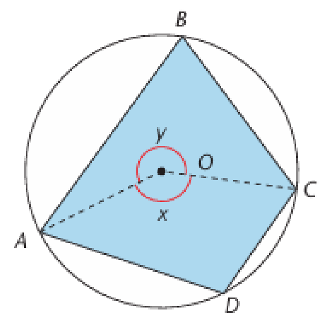
angle $ADC = \frac{1}{2}y$

$x + y = 360°$ The angles around a point add up to 360°.

angle ABC + angle $ADC = \frac{1}{2}(x + y)$

angle ABC + angle $ADC = 180°$

So the opposite angles of a cyclic quadrilateral are supplementary.

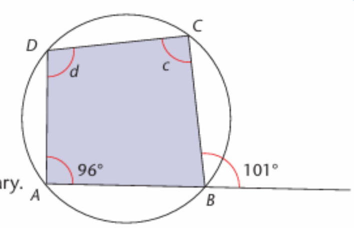


Example 38.5

Question

Find the sizes of angles c and d.

Give a reason for each step of your work.



Solution

$c = 180 - 96 = 84°$ Opposite angles of a cyclic quadrilateral are supplementary.

Angle $ABC = 180 - 101 = 79°$ Angles on a straight line add up to 180°.

$d = 180 - 79 = 101°$ Opposite angles of a cyclic quadrilateral are supplementary.

Note

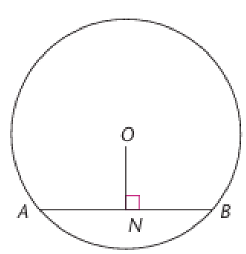
Two angles are *supplementary* if they add up to 180°.

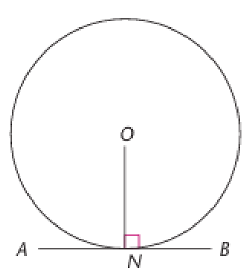
The tangent to a circle is at right angles to the radius at the point of contact

Earlier in this chapter, you saw that the perpendicular bisector of a chord passes through the centre of the circle.

So, the line from the centre to the midpoint of the chord is at right angles to the chord.

If you move the chord further from the centre it will eventually become a tangent and ON will, be a radius.



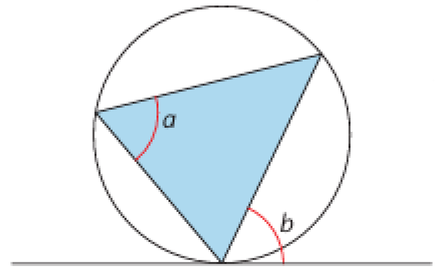


So, the tangent to a circle is at right angles to the radius at the point of contact

The alternate segment theorem

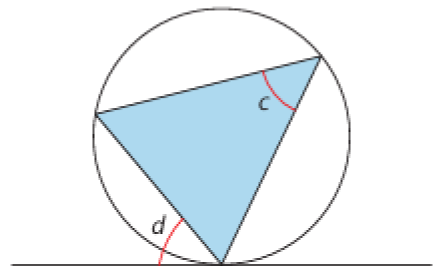
The angle between a tangent and a chord is equal to any angle made by that chord in the alternate segment of the circle

$a = b$

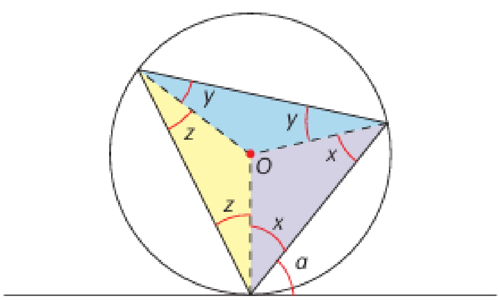


and

$c = d$



This can be proved as follows. O is the centre of the circle.



The dotted radii make three isosceles triangles with the base angles equal in each triangle.

$a = 90 - x$ The tangent is perpendicular to the radius.

$2x + 2y + 2z = 180$ Angles in a triangle add up to 180°.

So that $x + y + z = 90$

And $y + z = 90 - x$

Therefore $a = y + z$

So, the angle between the tangent and the chord, a, equals the angle in the alternate segment, $y + z$.

Example 38.6

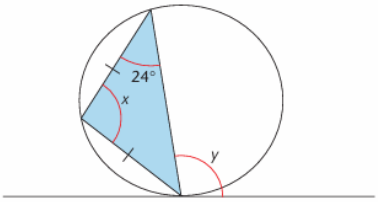
Question

Find the size of angle x and angle y.

Solution

$x = 180 - 2 \times 24 = 132°$ Angles at the base of an isosceles triangle are equal.

$y = 132°$ Alternate segment theorem.



Key points

- The perpendicular from the centre of a circle to a chord bisects the chord.
- Equal chords are equidistant from the centre of the circle.
- Tangents to a circle from an external point are equal in length.
- The angle at the centre of the circle is twice the angle at the circumference.
- The angle in a semi-circle is 90°.
- Angles in the same segment are equal.
- Opposite angles in a cyclic quadrilateral add up to 180°.
- The tangent to a circle is perpendicular to the radius.
- The angle between a tangent and a chord is equal to the angle in the alternate segment.

Chapter - 39 UNITS OF MEASURE

Basic units of length, mass and capacity

These are the connections between the metric units of length.

1 kilometre = 1000 metres	1 km = 1000 m
1 metre = 1000 millimetres	1 m = 1000 mm
1 metre = 100 centimetres	1 m = 100 cm
1 centimetre = 10 millimetres	1 cm = 10 mm

These are the connections between the metric units of mass

1 kilogram = 1000 grams	1 kg = 1000 g

When a volume is filled with liquid or gas, it is called the **capacity**.
These are the connections between the metric units of capacity.

1 litre = 1000 millilitres	1 l = 1000 ml

A millilitre is exactly the same as a cm^3 but is used for liquids rather than cm^3.

Example 39.1

Question

Misha is baking some cakes.

One recipe needs 1.6 kg of flour, another needs $\frac{1}{2}$ kg of flour.

a How much flour does he need altogether?

b He has a new 3 kg bag of flour.
How much will be left after he has made the cakes?

Solution

a When you have to add fractions and/or decimal parts of a kilogram, it is usually easier to change all the weights to grams.

Total = 1.6 kg + $\frac{1}{2}$ kg = 1600 g + 500 g
= 2100 g or 2.1 kg

b Amount of flour left = 3 kg – 2.1 kg = 3000 g – 2100 g
= 0.9 kg or 900 g

Area and volume measures

You can use the basic relationships between metric units of length to work out the relationships between metric units of area and volume.

For example:

$$1\,cm^2 = 1\,cm \times 1\,cm = 10\,mm \times 10\,mm = 100\,mm^2$$

$$1\,cm^3 = 1\,cm \times 1\,cm \times 1\,cm = 10\,mm \times 10\,mm \times 10\,mm = 1000\,mm^3$$

$$1\,m^2 = 1\,m \times 1\,m = 100\,cm \times 100\,cm = 10\,000\,cm^2$$

$$1\,m^3 = 1\,m \times 1\,m \times 1\,m = 100\,cm \times 100\,cm \times 100\,cm = 1\,000\,000\,cm^3$$

Note

Remember that volume and capacity units are related.

1 litre = $1000\,cm^3$

$1\,ml = 1\,cm^3$

Example 39.2

Question

Change these units.

a $5\,m^3$ to cm^3 **b** $5600\,cm^2$ to m^2

Solution

a $5\,m^3 = 5 \times 1\,000\,000\,cm^3 = 5\,000\,000\,cm^3$ Convert $1\,m^3$ to cm^3 and multiply by 5.

b $5600\,cm^2 = 5600 \div 10\,000\,m^2 = 0.56\,m^2$ To convert from m^2 to cm^2 you multiply, so to convert from cm^2 to m^2 you divide.

Note

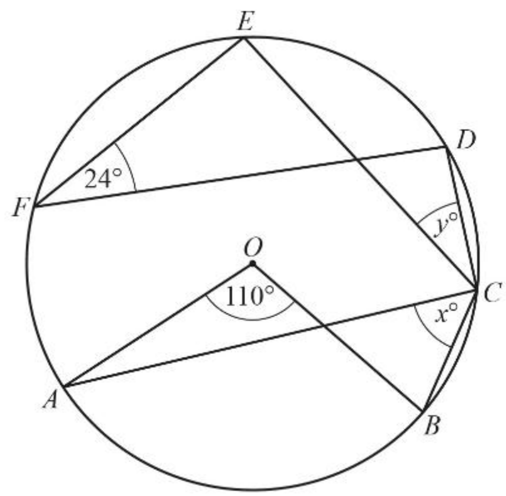
Make sure you have done the right thing by checking that your answer makes sense. If you had multiplied by 10000, you would have got $56\,000\,000\,m^2$, which is obviously a much larger area than $5600\,cm^2$.

Key points

- Everyday units of length are millimetres (mm), centimetres (cm), metres (m) and kilometres (km).
- There are 10 millimetres in a centimetre, 100 centimetres in a metre and 1000 metres in a kilometre.
- Everyday units of mass are kilograms (kg) and grams (g).
- There are 1000 grams in a kilogram.
- Everyday units of capacity are millilitres (ml) and litres (l).
- There are 1000 millilitres in a litre.
- A millilitre has the same volume as $1\,cm^3$.
- $1\,km^2 = 1\,000\,000\,m^2$. $1\,m^2 = 10\,000\,cm^2$. $1\,cm^2 = 100\,mm^2$.
- 1 litre = $1000\,cm^3$. $1\,ml = 1\,cm^3$.
- $1\,m^3 = 1\,000\,000\,cm^3 = 1000$ litres.

Revision questions

1.

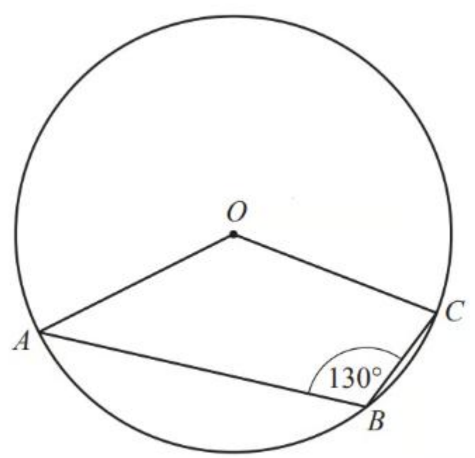


NOT TO SCALE

Points A, B, C, D, E and F lie on the circle, centre O.

Find the value of x and the value of y.

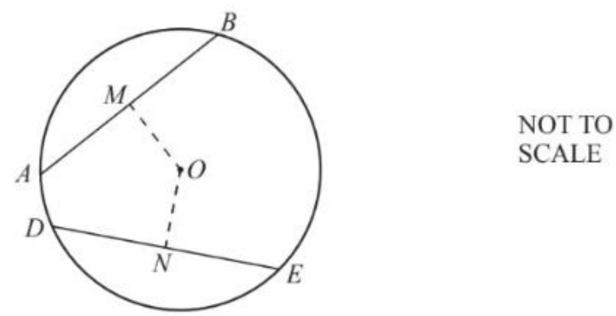
2.



NOT TO SCALE

A, B and C are points on the circle, centre O.
Find the obtuse angle AOC.

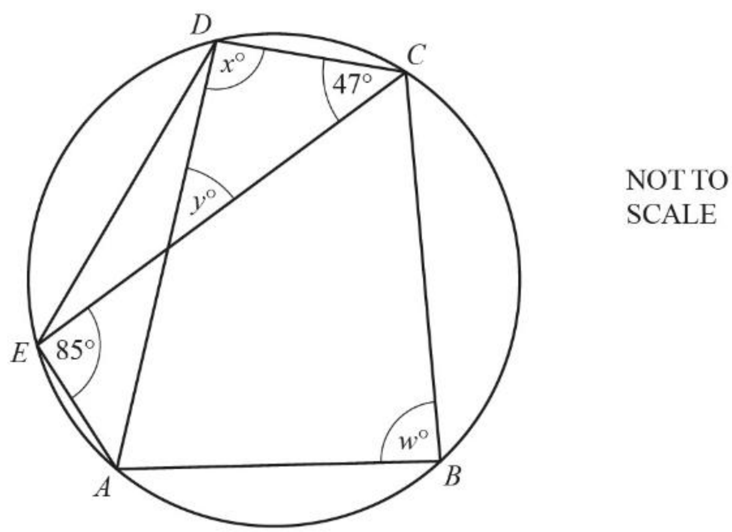
3.



The diagram shows a circle, centre O.
AB and DE are chords of the circle.
M is the mid-point of AB and N is the mid-point of DE.
$AB = DE = 9$ cm and $OM = 5$ cm.

Find ON.

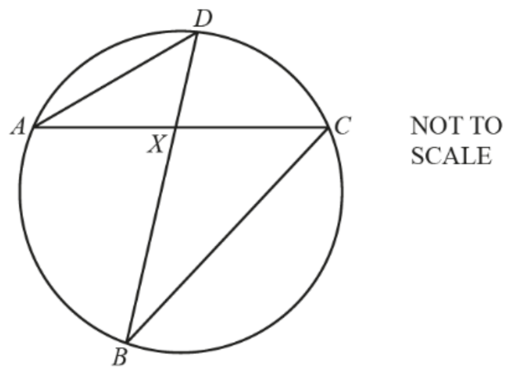
4.



The points A, B, C, D and E lie on the circumference of the circle.
Angle $DCE = 47°$ and angle $CEA = 85°$.

Find the values of w, x and y.

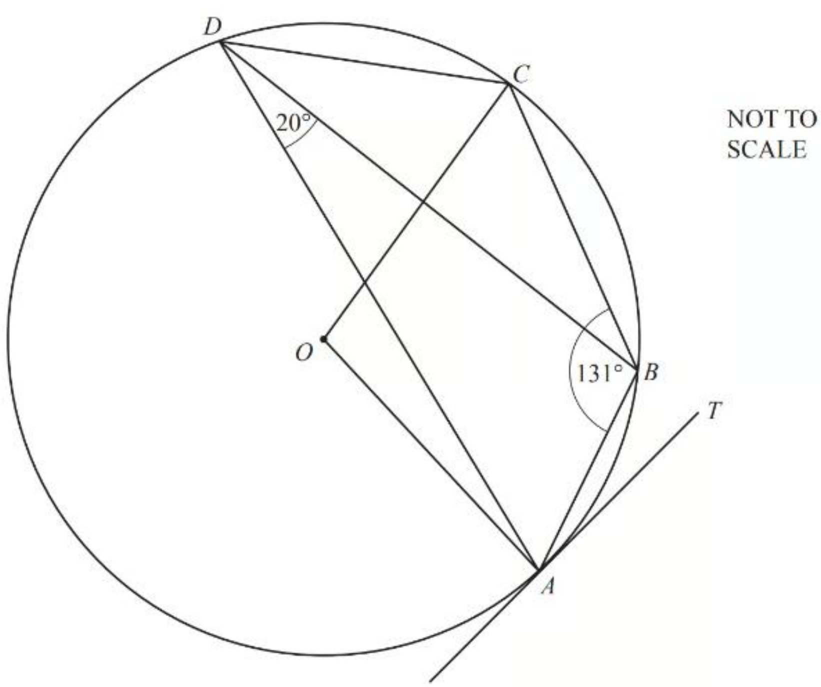
5.



A, B, C and D are points on the circumference of the circle.
AC and BD intersect at X.
Complete the statement.

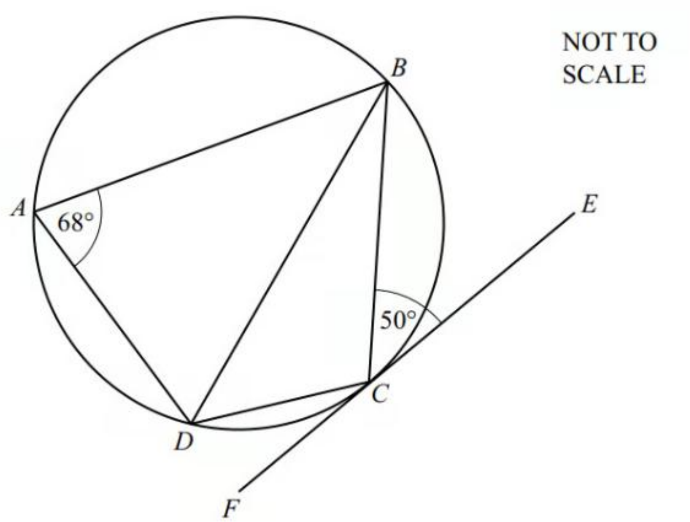
Triangle ADX is .... to triangle BCX.

6.



A, B, C and D lie on the circle, centre O.
Angle $ABC = 131°$.
Find angle ADC.

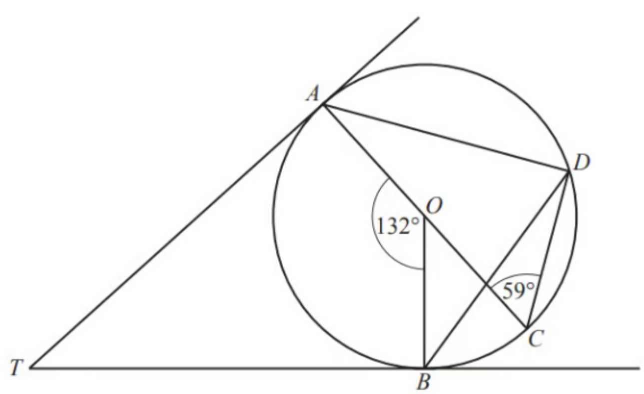
7.



A, B, C and D are points on the circumference of a circle.
EF is a tangent to the circle at C.
Angle $BAD = 68°$ and angle $BCE = 50°$.

Find angle CBD.
Give a geometrical property to explain each step of your working.

8.

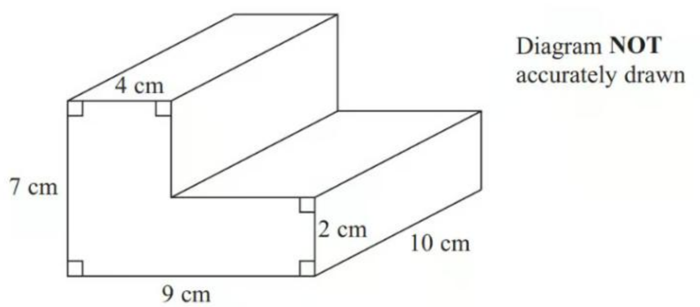


In the diagram, the points A, B, C and D lie on the circle, centre O.
TA and TB are tangents touching the circle at A and B respectively.

$A\hat{O}B = 132°$, $A\hat{C}D = 59°$ and AOC is a straight line.

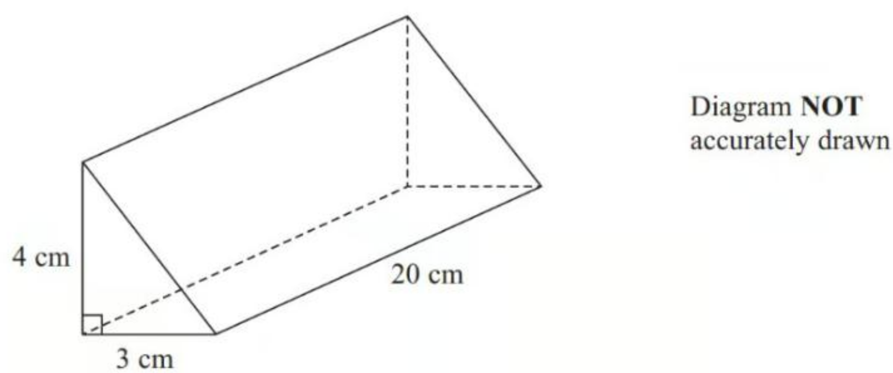
Find $A\widehat{T}B$.

9. The diagram shows a prism.



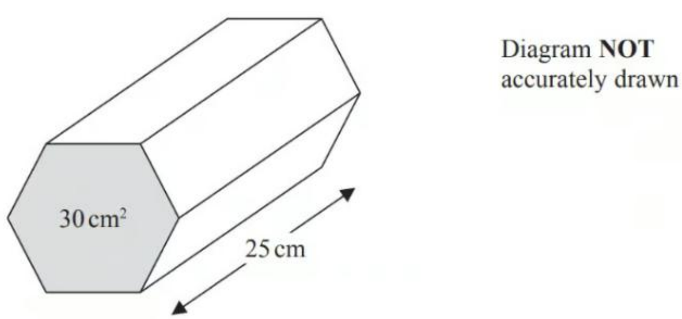
Work out the volume of the prism.

10. Here is a triangular prism.



Work out the volume of this triangular prism.

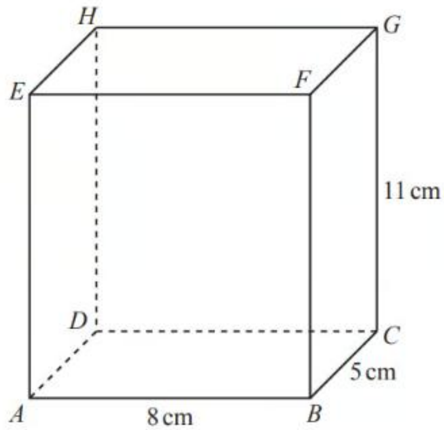
11. The diagram shows a prism.



The area of the cross section of the prism is 30 cm^2.
The length of the prism is 25 cm.

Work out the volume of the prism.

12.



NOT TO SCALE

ABCDEFGH is a cuboid.
$AB = 8$ cm, $BC = 5$ cm and $CG = 11$ cm.
Work out the volume of the cuboid.