Cambridge OL

Mathematics

CODE: (4024)

Chapter 42 and chapter 43

Transformations and vectors

Chapter 42 – Transformations

The language of transformations

When a shape is transformed, the original shape is called the object and the new shape is called the image.

A transformation maps the object on to the image.
If the vertices of an object are labelled A, B, C, …, the corresponding vertices of the image can be labelled A′, B′, C′ ….
Anything that stays the same when a transformation is performed is invariant.

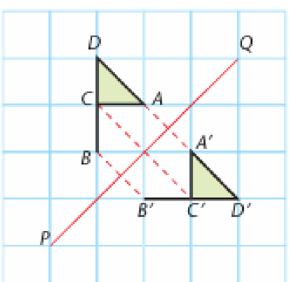
Reflection

Reflections are carried out in **mirror lines.**

In a **reflection**, corresponding points are the same distance from the mirror line but on the opposite side.

The object and the image are congruent, but the image is reversed. Points on the mirror lines are invariant points under a reflection.

In this diagram, the shape has been reflected in the line PQ.

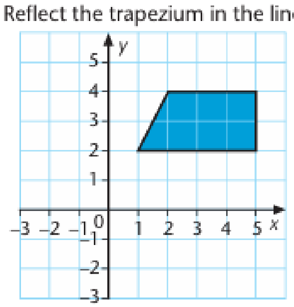


Instead of counting squares you could trace the shape and the mirror line, then turn the tracing paper over and line up the mirror line and your tracing of it.

Example 42.1

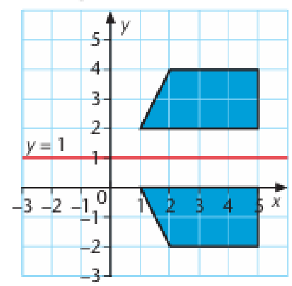
Question

Reflect the trapezium in the line $y = 1$.



Solution

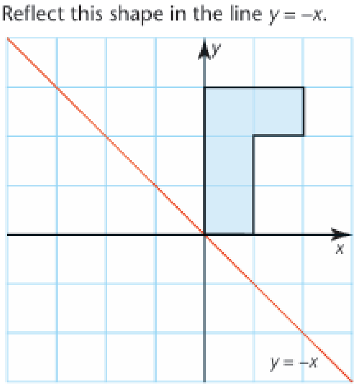
The diagram shows the reflection.



Example 42.2

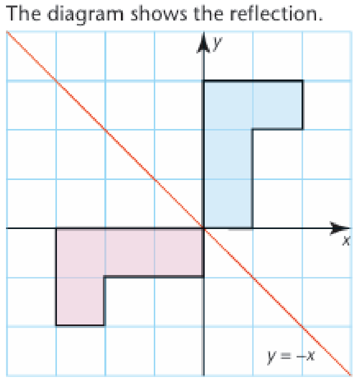
Question

Reflect this shape in the line $y = -x$.



Solution

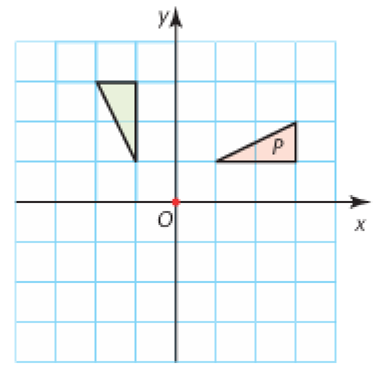
The diagram shows the reflection.



Note

When you have drawn a reflection in a sloping line, check it by turning the page so the mirror line is vertical. Then you can easily see if it has been reflected correctly.

Rotation



In a rotation you turn an object about a point. The point is called the centre of rotation. When an object is rotated, the object and its image are congruent. Unlike reflections, when an object is rotated, the object and image are the same way round. In this diagram, shape P has been rotated 90° anticlockwise about O.

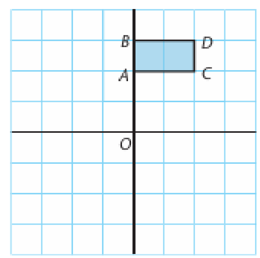
To rotate a shape you need to know three things

- the angle of rotation
- the direction of rotation
- the centre of rotation.

Example 42.3

Question

Rotate this shape 270° anticlockwise about *O*.



Solution

You can think of 270° anticlockwise as being the same as 90° clockwise.

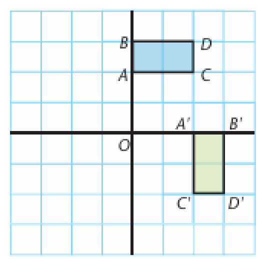
You can draw the rotation by counting squares.

A is 2 squares above *O* so its image, *A′*, is 2 squares to the right.

B is 3 squares above *O* so its image, *B′*, is 3 squares to the right.

C is 2 squares above and 2 to the right of *O* so its image, *C′*, is 2 to the right and 2 below *O*.

D is 3 squares above and 2 to the right of *O* so its image, *D′*, is 3 to the right and 2 below *O*.



Alternatively, you could do the rotation using tracing paper.

Trace the shape.

Hold the centre of rotation still with the point of a pin or a pencil and rotate the paper through a quarter-turn clockwise.

Use another pin or the point of your compasses to prick through the corners.

Join the pin holes to form the image.

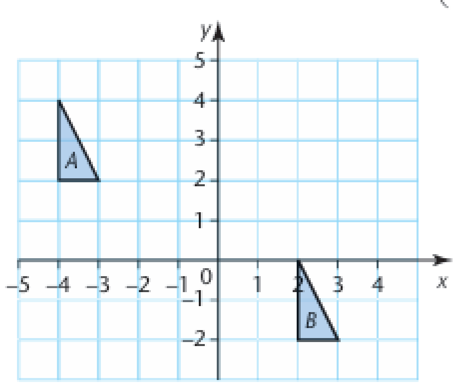
Translation

In a translation, every point on the object moves the same distance in the same direction as every other point. The object and the image are congruent.

In this diagram, triangle A maps on to triangle B.

Each vertex of the triangle moves 6 squares to the right and 4 squares down.

The translation is given by the vector $\begin{pmatrix} 6 \\ -4 \end{pmatrix}$.



Note

Take care with the counting.

Choose a point on both the object and the image and count the units from one to the other.

Note

You will learn about vectors in Chapter 43.

Enlargement

In an enlargement of a shape, the image is the same shape as the object, but it is larger or smaller.

The object and image are not congruent, because they are different sizes, but they are similar. An enlargement is drawn from a given point, the centre of enlargement.

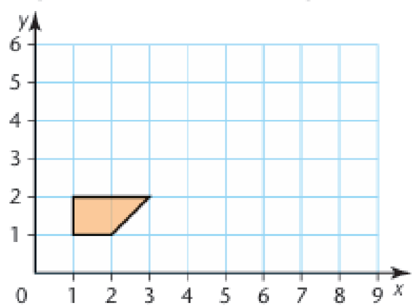
For a ‘2 times enlargement’, each point on the image must be twice as far from the centre as the corresponding point on the object. If the centre is on the object, that point will not move and the enlargement will overlap the original shape.

The number used to multiply the lengths for the enlargement is the scale factor.

Example 42.4

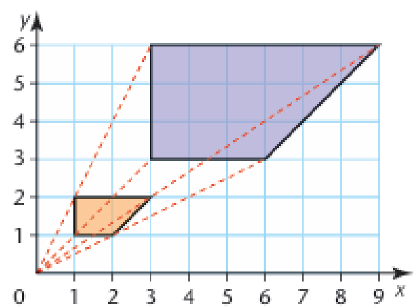
Question

Enlarge this shape by scale factor 3 with the origin as the centre of enlargement.



Solution

Scale factor 3 means this is a ‘3 times enlargement’, so each point in the image is three times as far from the origin as the corresponding point in the original shape.



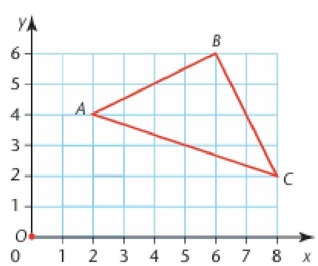
Enlargement with a fractional scale factor

The technique for drawing an enlargement with a fractional scale factor is the same as for a positive integral scale factor. You can see how this is done in the following example.

Example 42.5

Question

Draw an enlargement of triangle *ABC* with centre *O* and scale factor $\frac{1}{2}$.



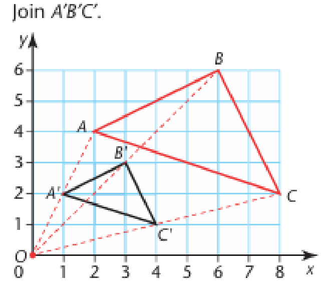
Solution

Draw lines from *O* to *A*, *B* and *C*.

By counting squares, mark *A′* on *OA* so that *A′* is the midpoint of *OA*.

Similarly mark *B′* and *C′* at the midpoints of *OB* and *OC*.

Join *A′B′C′*.



A′B′C′ is the required enlargement with scale factor $\frac{1}{2}$.

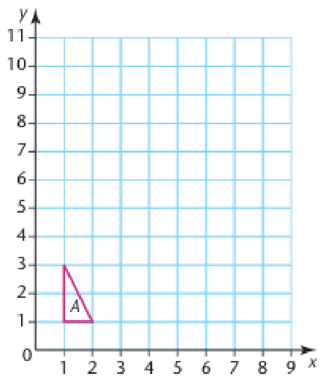
Check that the lengths of *A′B′*, *B′C′* and *A′C′* are $\frac{1}{2}$ the lengths of the corresponding lengths on the original triangle.

Enlargement with a negative scale factor

When the scale factor of an enlargement is negative, the image is on the opposite side of the centre of enlargement from the object and the image is inverted.

Example 42.6

Question



Enlarge triangle *A* with scale factor −2 and centre (3, 4).

Solution

Plot the centre of enlargement, *O*, at (3, 4).

Draw the line *AO* and extend it. Measure the length *OA* and mark the point *A′* so that *OA′* = 2 × *OA*.

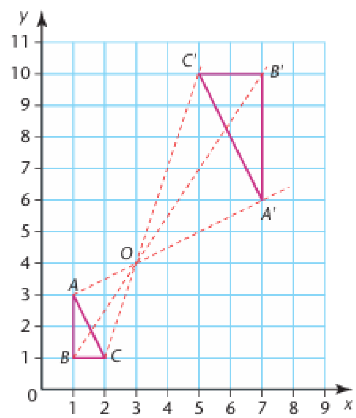
Note that *A′* is on the opposite side of *O* to *A*.

Or, by counting squares, *O* is 2 to the right and one above *A*, so *A′* is 4 to the right and 2 above *O*.

Do the same for *BO* to give point *B′* and *CO* to give point *C′*.

A′B′C′ is the required image of *ABC*.

Check that the sides of triangle *A′B′C′* are twice as long as the corresponding sides of triangle *ABC*.



Recognising and describing transformations

You should know how to describe fully a transformation if given the object and the image. You must first state which type of transformation it is. Trace the object. If you have to turn it over to fit on the image, the transformation is a reflection.

If you have to turn the tracing round but not turn it over, the transformation is a rotation. If you just slide the tracing without turning over or turning round, the transformation is a translation.

If the object is not the same size as the image, the transformation is an enlargement. You then need to complete the description.

The extra information required is shown in this table.

Transformation	Extra information required
Reflection	The equation of the mirror line or a statement that it is the *x*-axis or the *y*-axis
Rotation	Angle Direction Centre
Translation	Column vector
Enlargement	Scale factor Centre

Note

When describing fully a *single* transformation, do not give a combination of transformations.

Reflection

Example 42.7

Question

Describe fully the single transformation that maps triangle *ABC* on to triangle *A′B′C′*.

Solution

You can check that the transformation is a reflection by using tracing paper.

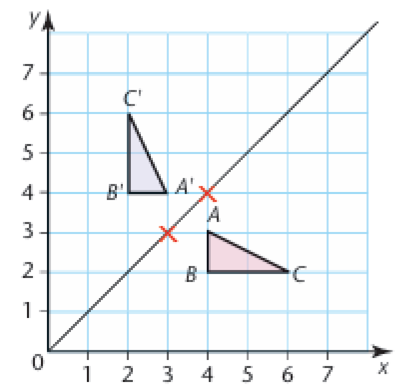
To find the mirror line, put a ruler between two corresponding points (*B* and *B′*) and mark a point halfway between them, at (3, 3).

Repeat this for two other corresponding points (*C* and *C′*). The midpoint is (4, 4).

Join the two midpoints to find the mirror line. The mirror line is $y = x$.

The transformation is a reflection in the line $y = x$.

Again, the result can be checked using tracing paper.

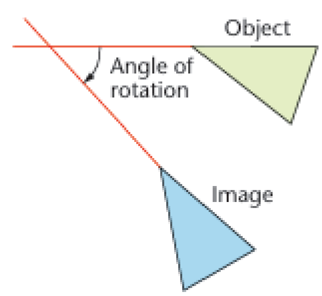


Rotation

To find the angle of rotation, find a pair of sides that correspond in the object and the image. Measure the angle between them.

You may need to extend both of the sides to do this. If the centre of rotation is not on the object, its position may not be obvious.

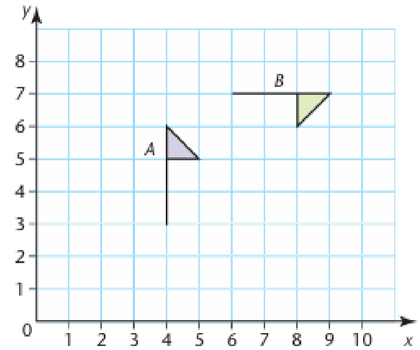
However, you can usually find it, either by counting squares or using tracing paper. Always remember to state whether the rotation is clockwise or anticlockwise.



Example 42.8

Question

Describe fully the single transformation that maps flag *A* on to flag *B*.



Solution

You have to turn the shape round but you don't turn it over, so the transformation is a rotation.

The angle is 90° clockwise.

You may need to make a few trials, using tracing paper and a compass point centred on different points, to find that the centre of rotation is (7, 4).

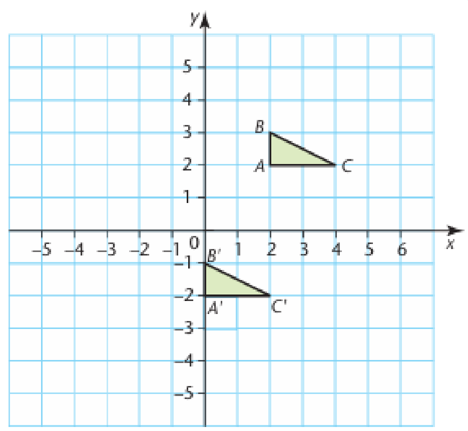
So the transformation that maps flag *A* on to flag *B* is a rotation of 90° clockwise, with centre of rotation (7, 4).

Translation

Example 42.9

Question

Describe fully the single transformation that maps shape *ABC* on to shape *A′B′C′*.



Solution

The shape stays the same way up and is the same size so the transformation is a translation.

A moves from (2, 2) to (0, –2). This is a movement of 2 to the left and 4 down.

The transformation is a translation by the vector $\begin{pmatrix} -2 \\ -4 \end{pmatrix}$.

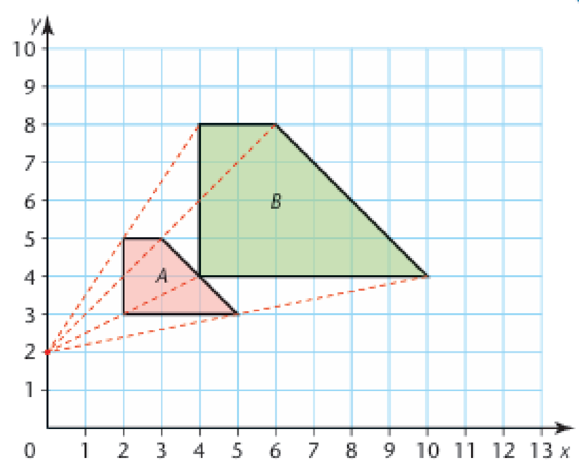
Enlargement

Example 42.10

Question

Describe fully the transformation that maps

a *A* on to *B* **b** *B* on to *A*.



Solution

By measuring the sides you can see that shape *B* is twice as big as shape *A*.

To find the centre of enlargement, you join corresponding vertices of the two shapes and extend them until they cross.

The point where they cross is the centre of enlargement.

The centre of enlargement is (0, 2).

a The transformation that maps shape *A* on to shape *B* is an enlargement, scale factor 2, centre (0, 2).

b The transformation that maps shape *B* on to shape *A* is an enlargement, scale factor $\frac{1}{2}$, centre (0, 2).

Combining transformations

If a transformation is followed by another transformation, the result is sometimes equivalent to a third transformation.

> **Note**
>
> It is very easy to assume PQ means P followed by Q.
>
> It does not. It means Q followed by P.
>
> This notation is like that used for functions.

Example 42.11

Question

Describe fully the single transformation equivalent to a rotation of 90° clockwise about the origin, followed by a reflection in the y-axis.

Solution

Choose a simple shape like triangle A to start with.

Rotating triangle A 90° clockwise about the origin gives triangle B.

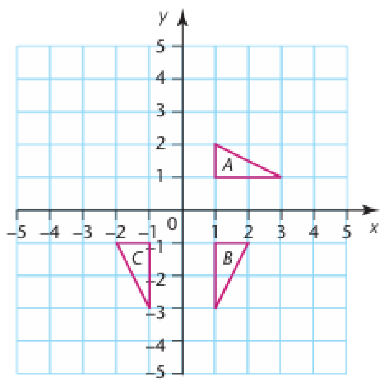
Reflecting triangle B in the y-axis gives triangle C.

You need to find the single transformation that will map triangle A on to triangle C.

If you trace triangle A, you will need to turn the tracing over to fit it on to triangle C, so the transformation must be a reflection.

The mirror line is the $y = -x$ line.

So the single transformation is a reflection in the line $y = -x$.



> **Note**
>
> Make sure you carry out the transformations in the order stated in the question. It usually makes a difference.
>
> When describing a single transformation, do not give a combination of transformations.

Example 42.12

Question

E is the transformation: an enlargement, scale factor 3, centre (3, 3).

T is the transformation: a translation by the vector $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

Describe fully the single transformation that is equivalent to E followed by T.

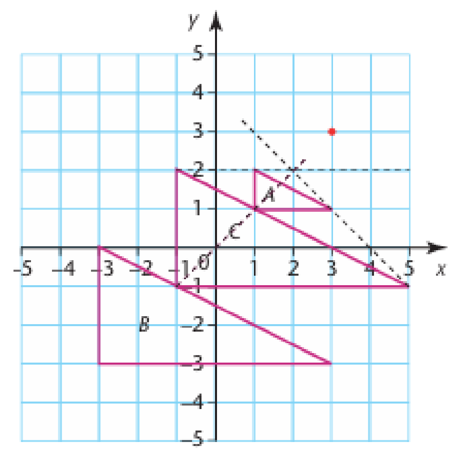
Solution

Again, start with a simple object like triangle A.

Enlarge triangle A with scale factor 3, centre (3, 3).

This maps triangle A on to triangle B.

Then translate triangle B by the vector $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$.



This maps triangle B on to triangle C.

Clearly the equivalent transformation to E followed by T is still an enlargement with scale factor 3. It just remains to find the centre of enlargement.

Draw lines through the corresponding vertices of triangle A and triangle C. These intersect at (2, 2).

So the single transformation is an enlargement, scale factor 3, with centre (2, 2).

Key points

- When a shape is reflected, rotated or translated, it stays exactly the same shape. The shape and its image are congruent.
- In a translation, every point on the shape moves the same distance, in the same direction.
- In a rotation, all the points move through the same angle about the same centre.
- In a reflection, each point on the shape and the corresponding point on its image are the same distance from the mirror line, but on opposite sides.
- When a shape is enlarged by, for example, a scale factor of 2, from a given centre, each point on the enlargement will be 2 times as far away from the centre as the corresponding point on the original shape.
- A shape and its enlargement are similar.
- To find the scale factor of an enlargement, divide the length of a side of the image by the length of the corresponding side of the object.
- To find the centre of enlargement, join the corresponding corners of the two shapes with straight lines using a ruler and extend the lines until they cross. The point where they cross is the centre of enlargement.
- When the scale factor is between 0 and 1, such as $\frac{1}{2}$, the image is smaller than the object. However, the transformation is still called an enlargement.
- When the scale factor is negative, the image is on the opposite side of the centre of enlargement from the object, and the image is inverted.
- When describing a transformation, first give the name of the transformation. Then, give the extra information required.
- When combining transformations, make sure you carry out the operations in the correct order.

Chapter 43 – Vectors

Vectors and translations A transformation that moves points in a given direction for a given distance is called a translation.

The distance and direction that a shape moves in a translation can be written as a column vector. The top number tells you how far the shape moves across, or in the x-direction.

The bottom number tells you how far the shape moves up or down, or in the y-direction

A positive top number is a move to the right.

A negative top number is a move to the left.

A positive bottom number is a move up.

A negative bottom number is a move down.

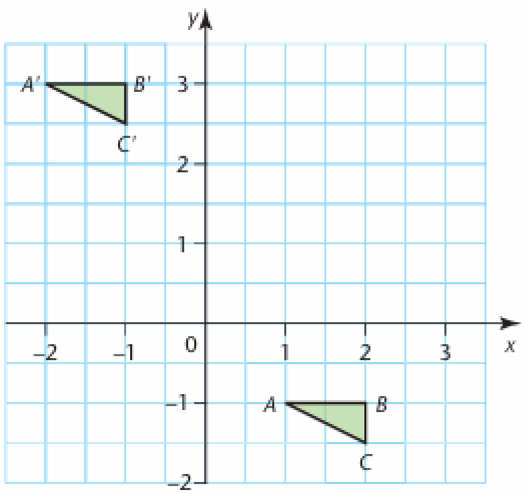
A translation of 3 to the right and 2 down is written as $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

Example 43.1

Question

Translate the triangle ABC by $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$.

Solution



$\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ means move 3 units left and 4 units up.

Point A moves from (1, –1) to (–2, 3).

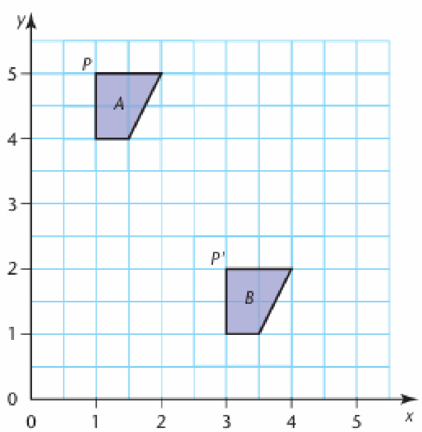
Point B moves from (2, –1) to (–1, 3).

Point C moves from (2, –1.5) to (–1, 2.5).

Example 43.2

Question

Describe fully the single transformation that maps shape A on to shape B.



Note

Try not to confuse the words *transformation* and *translation*.

Transformation is the general name for all changes made to shapes.

Translation is the particular transformation where all points of an object move the same distance in the same direction.

Solution

It is clearly a translation as the shape stays the same way up and the same size.

To find the movement, choose one point on the original shape and the image and count the units moved.

For example, P moves from (1, 5) to (3, 2). This is a movement of 2 to the right and 3 down.

The transformation is a translation of $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

Note

You must state that the transformation is a translation and give the column vector.

Vector notation

You have already met directed numbers; these are numbers which go in either a positive or a negative direction. Vectors, however, can go in any direction. They have two parts, a magnitude (length) and a direction. They are very useful for the study of motion.

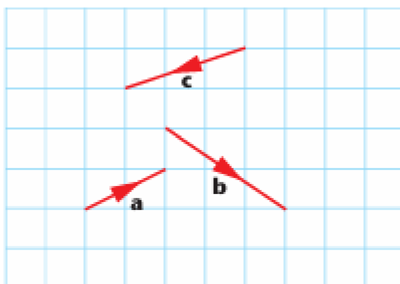
Vectors are usually written in one of three ways.

- The most common notation for a vector is a column vector, for example $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$, meaning 3 in the positive x-direction and 2 in the positive y-direction.

  These are similar to coordinates, except written in a column, and they can start anywhere (not just at the origin).
- $\overrightarrow{AB}$ meaning the vector starts at A and finishes at B.
- **a**. This is the algebraic form of a vector.

Column vectors

If a vector is drawn on a grid then it can be described by a column vector $\begin{pmatrix} x \\ y \end{pmatrix}$, where x is the length across to the right and y is the length upwards.



In the diagram, $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$.

Example 43.3

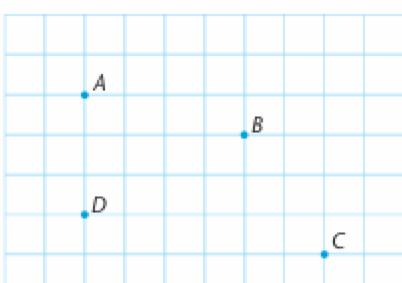
Question

Write down these column vectors.

a The vector that maps point A to point B.

b The vector $\overrightarrow{CD}$.

c The vector $\overrightarrow{DC}$.



Note

Students often make an error of 1 when working out the values for the vector.

Take care with the counting.

Solution

a $\overrightarrow{AB} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ **b** $\overrightarrow{CD} = \begin{pmatrix} -6 \\ 1 \end{pmatrix}$ **c** $\overrightarrow{DC} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$

Example 43.4

Question

Find the column vector that maps (–1, 6) on to (5, 2).

Solution

The x-coordinate has changed from –1 to 5 so the increase is 6.

The y-coordinate has changed from 6 to 2 so the decrease is 4.

The vector is $\begin{pmatrix}6\\-4\end{pmatrix}$.

Note

As you can see, it is not necessary to plot the points to do this question.

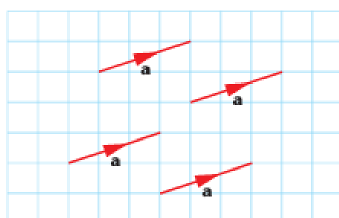
However, you may prefer to do so.

General vectors

A vector has both length and direction but can be in any position. The vector going from A to B can be labelled $\overrightarrow{AB}$ or it can be given a letter **a**, in bold type.

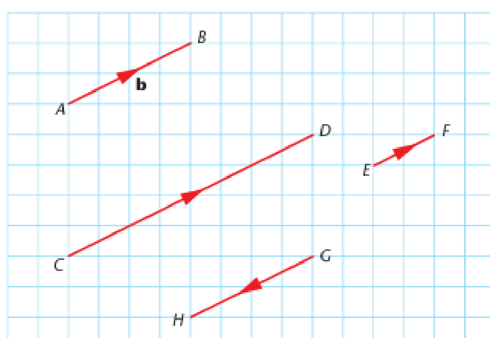
When you write a vector, you must indicate that the quantity is a vector in some definite way. You can use an arrow, for example, $\overrightarrow{AB}$, to show the vector that goes from A to B or you can use underlining, for example, $\underline{a}$, to show a vector that in print is bold, **a**.

All four lines drawn below are of equal length and go in the same direction and they can all be called **a**.



Look at the diagram below.

$\overrightarrow{AB} = \mathbf{b}$.



The line CD is parallel to AB and twice as long so $\overrightarrow{CD} = 2\mathbf{b}$.

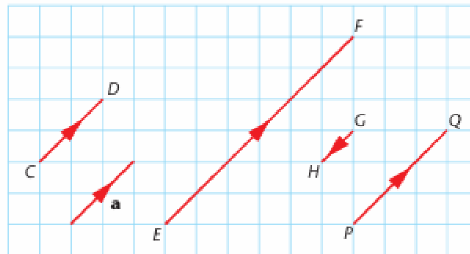
EF is parallel to AB and half the length so $\overrightarrow{EF} = \frac{1}{2}\mathbf{b}$.

GH is parallel and equal in length to AB but in the opposite direction so $\overrightarrow{GH} = -\mathbf{b}$.

Example 43.5

Question

For the diagram, write down the vectors $\overrightarrow{CD}$, $\overrightarrow{EF}$, $\overrightarrow{GH}$ and $\overrightarrow{PQ}$ in terms of **a**.



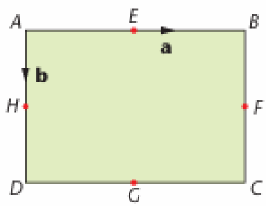
Solution

$\overrightarrow{CD} = \mathbf{a}$, $\overrightarrow{EF} = 3\mathbf{a}$, $\overrightarrow{GH} = -\frac{1}{2}\mathbf{a}$, $\overrightarrow{PQ} = \frac{3}{2}\mathbf{a}$

Example 43.6

Question

ABCD is a rectangle and *E*, *F*, *G*, *H* are the midpoints of the sides.
$\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$.



Write the vectors $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{AE}$, $\overrightarrow{AH}$, $\overrightarrow{EG}$, $\overrightarrow{CF}$ and $\overrightarrow{FH}$ in terms of **a** or **b**.

Solution

$\overrightarrow{BC} = \mathbf{b}$, $\overrightarrow{CD} = -\mathbf{a}$, $\overrightarrow{AE} = \frac{1}{2}\mathbf{a}$, $\overrightarrow{AH} = \frac{1}{2}\mathbf{b}$, $\overrightarrow{EG} = \mathbf{b}$, $\overrightarrow{CF} = -\frac{1}{2}\mathbf{b}$, $\overrightarrow{FH} = -\mathbf{a}$

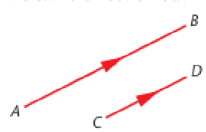
Multiplying a vector by a scalar

A quantity that has magnitude but not direction is called a scalar.

Multiplying a vector by a scalar produces a vector in the same direction but longer by a factor equal to the scalar.

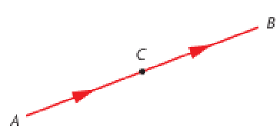
If you know that $\overrightarrow{AB} = k\overrightarrow{CD}$, you can conclude that

- $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$
- $\overrightarrow{AB}$ is k times the length of $\overrightarrow{CD}$.



If you know that $\overrightarrow{AB} = k\overrightarrow{AC}$ and there is a common point *A*, you can conclude that

- *A*, *B* and *C* are in a straight line
- $\overrightarrow{AB}$ is k times the length of $\overrightarrow{AC}$.



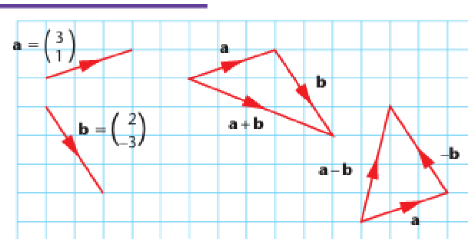
Addition and subtraction of column vectors

In the diagram, $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

You can see that

$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

and $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.



So, to add or subtract column vectors, you add or subtract the components.

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix} \text{ and } \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a-c \\ b-d \end{pmatrix}$$

Example 43.7

Question

Given $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, work out these.

a $2\mathbf{a}$ **b** $\mathbf{a} + 2\mathbf{b}$ **c** $\mathbf{a} - \mathbf{b} + \mathbf{c}$ **d** $2\mathbf{a} + \mathbf{b} - \mathbf{c}$ **e** $\frac{1}{2}\mathbf{a}$

Solution

a $2\mathbf{a} = 2 \times \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$

b $\mathbf{a} + 2\mathbf{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + 2 \times \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$

c $\mathbf{a} - \mathbf{b} + \mathbf{c} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

d $2\mathbf{a} + \mathbf{b} - \mathbf{c} = 2 \times \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$

e $\frac{1}{2}\mathbf{a} = \frac{1}{2} \times \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 0.5 \end{pmatrix}$

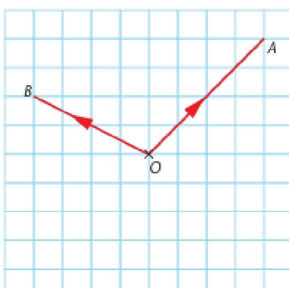
Note

When adding and subtracting column vectors, be very careful with the signs as most errors are made in that way.

Position vectors

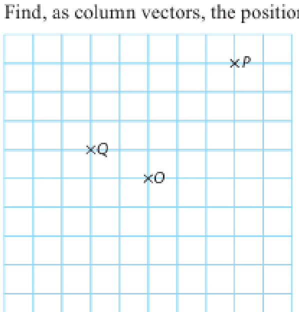
The vector $\overrightarrow{OA}$ is the **position vector** of A in relation to O.

Similarly $\overrightarrow{OB}$ is the position vector of B in relation to O.



Exercise 43.4

1 Find, as column vectors, the position vectors relative to O of P, Q and R.



2 A is the point (−2, 1), B is the point (4, 3) and C is the point (7, 4).

a What are the position vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$?

b Work out these column vectors.

i $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$

c What can you say about A, B and C?

3 A is the point (2, 1), B is the point (4, 4), C is the point (7, 4) and D is the point (3, −2).

a What are the position vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$?

b Work out these column vectors.

i $\overrightarrow{AB}$ **ii** $\overrightarrow{CD}$

c What can you say about AB and CD?

The magnitude of a vector

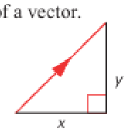
The magnitude of a vector is its length.

The magnitude of the vector $\overrightarrow{AB}$ is written as $|\overrightarrow{AB}|$.

The magnitude of the vector **a** is written as $|\mathbf{a}|$.

Pythagoras' theorem can be used to calculate the magnitude of a vector.

The **magnitude** of the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is $\sqrt{x^2 + y^2}$.



Example 43.8

Question

$\overrightarrow{CD} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

Find $|\overrightarrow{CD}|$.

Solution

$|\overrightarrow{CD}| = \sqrt{5^2 + (-2)^2}$

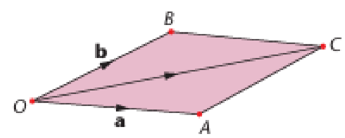
$= \sqrt{25 + 4}$

$= \sqrt{29}$

= 5.39 to 3 significant figures.

Vector geometry

In the diagram, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $OACB$ is a parallelogram.



AC is parallel and equal to OB, so $AC = \mathbf{b}$.

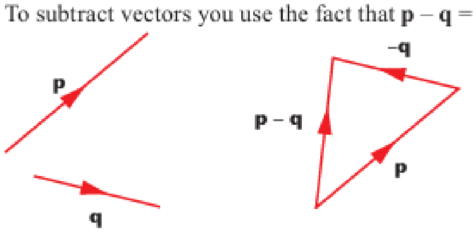
$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \mathbf{a} + \mathbf{b}$.

You can also see in the diagram that $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = \mathbf{b} + \mathbf{a}$.

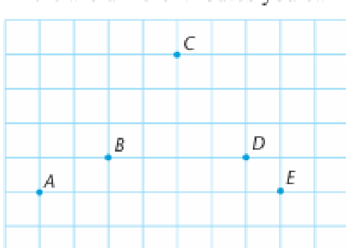
This shows that $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.

The vectors can be added in either order.

To subtract vectors you use the fact that $\mathbf{p} - \mathbf{q} = \mathbf{p} + (-\mathbf{q})$.



There are different routes you can use to get from A to E on this diagram.



For instance $\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CE} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$.

Whichever route you use, adding the column vectors gives the same result, $\overrightarrow{AE} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$.

The fact that the result is the same is a very important rule which, together with your knowledge of how to multiply a vector by a scalar, is used to find vectors in geometrical diagrams.

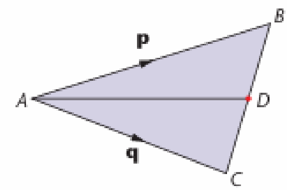
Example 43.9

Question

In the triangle ABC, $\overrightarrow{AB} = \mathbf{p}$, $\overrightarrow{AC} = \mathbf{q}$ and D is the midpoint of BC.

Write these vectors in terms of **p** and **q**.

a $\overrightarrow{BC}$ **b** $\overrightarrow{BD}$ **c** $\overrightarrow{AD}$



Solution

a $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$

$= -\mathbf{p} + \mathbf{q}$

$= \mathbf{q} - \mathbf{p}$

b $\overrightarrow{BD} = \frac{1}{2}\overrightarrow{BC}$

$= \frac{1}{2}(\mathbf{q} - \mathbf{p})$

c $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD}$

$= \mathbf{p} + \frac{1}{2}(\mathbf{q} - \mathbf{p})$

$= \mathbf{p} + \frac{1}{2}\mathbf{q} - \frac{1}{2}\mathbf{p}$

$= \frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$

$= \frac{1}{2}(\mathbf{p} + \mathbf{q})$

Example 43.10

Question

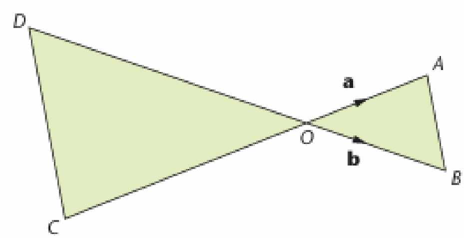
In this diagram, $OC = 2 \times OA$ and $OD = 2 \times OB$.

$\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

a Write these vectors in terms of **a** and **b**.

i $\overrightarrow{OC}$ **ii** $\overrightarrow{OD}$ **iii** $\overrightarrow{AB}$ **iv** $\overrightarrow{DC}$

b What does this show about the lines AB and DC?



Solution

a **i** OC is on the same line as OA, in the opposite direction and twice as long.

$\overrightarrow{OC} = -2 \times \overrightarrow{OA} = -2\mathbf{a}$

ii By the same reasoning as above, $\overrightarrow{OD} = -2 \times \overrightarrow{OB} = -2\mathbf{b}$.

iii $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$

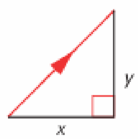
iv $\overrightarrow{DC} = \overrightarrow{DO} + \overrightarrow{OC} = 2\mathbf{b} - 2\mathbf{a} = 2(\mathbf{b} - \mathbf{a})$

b The vector for DC is twice the vector for AB.

So AB and DC are parallel and DC is twice as long as AB.

Key points

- A translation moves every point on an object the same distance, in the same direction.
- You can use a column vector to describe a translation.
  Translate by $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$ means move 4 units left and 2 units up.
- A vector has magnitude and direction but can start at any point.
- Multiplying a vector by a scalar (ordinary number) changes its length but not its direction.
- If you know that $\overrightarrow{AB} = k \times \overrightarrow{CD}$, then you know that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel and that the length AB is k times the length CD.
- If you know that $\overrightarrow{AB} = k \times \overrightarrow{AC}$, then you know that A, B and C are in a straight line (collinear) and that the length of AB is k times the length of AC.
- To multiply a column vector by a scalar, multiply each component by the scalar.
- To add or subtract column vectors, add or subtract the components.
- The vector $\overrightarrow{OA}$ is the position vector of A in relation to O.
- The magnitude of a vector is its length.
- The magnitude of the vector $\overrightarrow{AB}$ is written as $|\overrightarrow{AB}|$.
- The magnitude of the vector $\mathbf{a}$ is written as $|\mathbf{a}|$.
- Pythagoras' theorem can be used to calculate the magnitude of a vector.
  The magnitude of the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is $\sqrt{x^2 + y^2}$.



- In vector geometry, $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$. The vectors can be added in either order.

Revision questions

1.

$$\overrightarrow{GH} = \frac{5}{6}(2\mathbf{p} + \mathbf{q}) \qquad \overrightarrow{JK} = \frac{5}{18}(2\mathbf{p} + \mathbf{q})$$

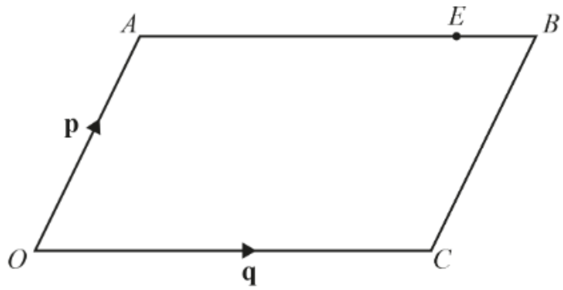
Write down two facts about the geometrical relationship between the vectors $\overrightarrow{GH}$ and $\overrightarrow{JK}$.

2.

$\overrightarrow{XY} = 3\mathbf{a} + 2\mathbf{b}$ and $\overrightarrow{ZY} = 6\mathbf{a} + 4\mathbf{b}$.

Write down two statements about the relationship between the points X, Y and Z.

3.

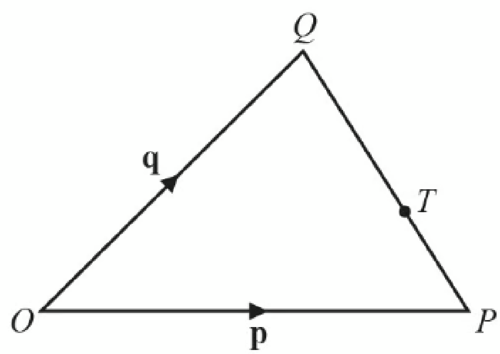


NOT TO SCALE

a parallelogram.
$\overrightarrow{OA} = \mathbf{p}$ and $\overrightarrow{OC} = \mathbf{q}$.
E is the point on AB such that $AE : EB = 3 : 1$.

Find $\overrightarrow{OE}$, in terms of $\mathbf{p}$ and $\mathbf{q}$, in its simplest form.

4.

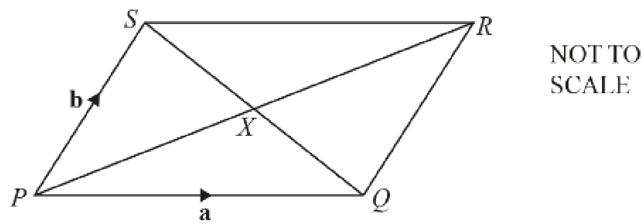


NOT TO SCALE

O is the origin, $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OQ} = \mathbf{q}$.
$QT : TP = 2 : 1$

Find the position vector of T.
Give your answer in terms of $\mathbf{p}$ and $\mathbf{q}$, in its simplest form.

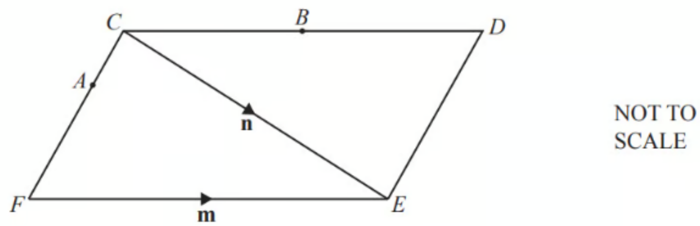
5.



$PQRS$ is a parallelogram with diagonals PR and PR intersecting at X.
$\overrightarrow{PQ} = \mathbf{a}$ and $\overrightarrow{PS} = \mathbf{b}$.

Find $\overrightarrow{QX}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
Give your answer in its simplest form.

6.



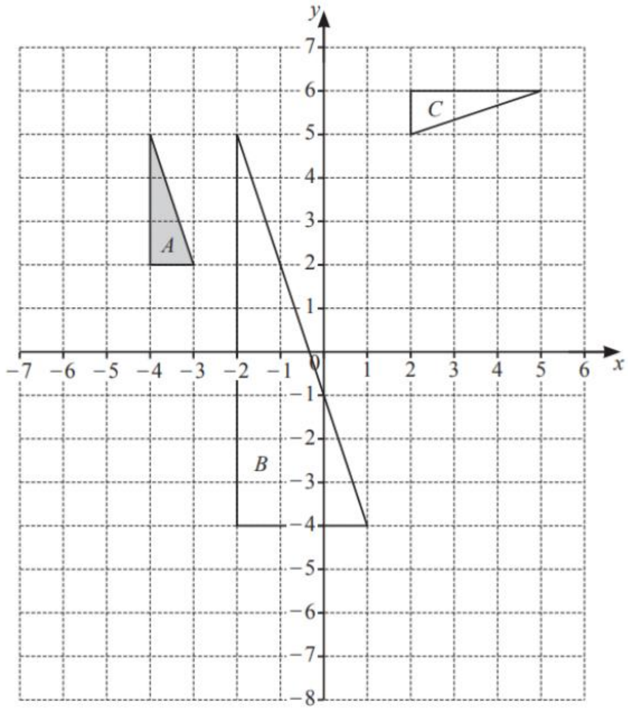
The diagram shows a parallelogram $CDEF$.
$\overrightarrow{FE} = \mathbf{m}$ and $\overrightarrow{CE} = \mathbf{n}$.
B is the midpoint of CD.
$FA = 2AC$

Find an expression, in terms of $\mathbf{m}$ and $\mathbf{n}$, for $\overrightarrow{AB}$.
Give your answer in its simplest form.

7.

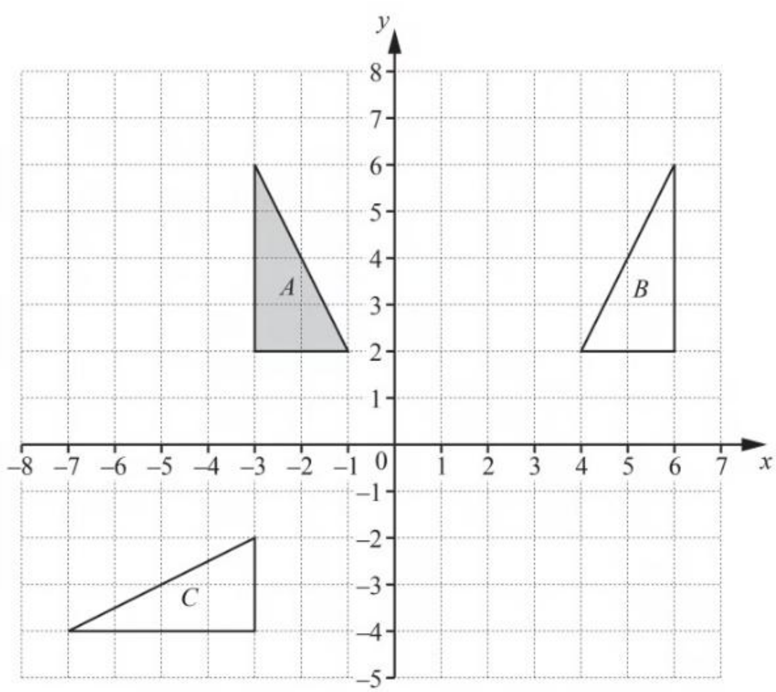
A line joins $A(1, 3)$ to $B(5, 8)$.
The line AB is transformed to the line PQ.
Find the co-ordinates of P and the co-ordinates of Q after AB is translated by the vector $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$.

8.



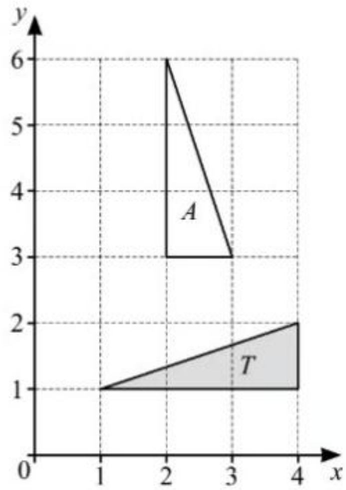
Draw the image of shape *A* after a translation by the vector $\begin{pmatrix} 8 \\ -6 \end{pmatrix}$.

9.



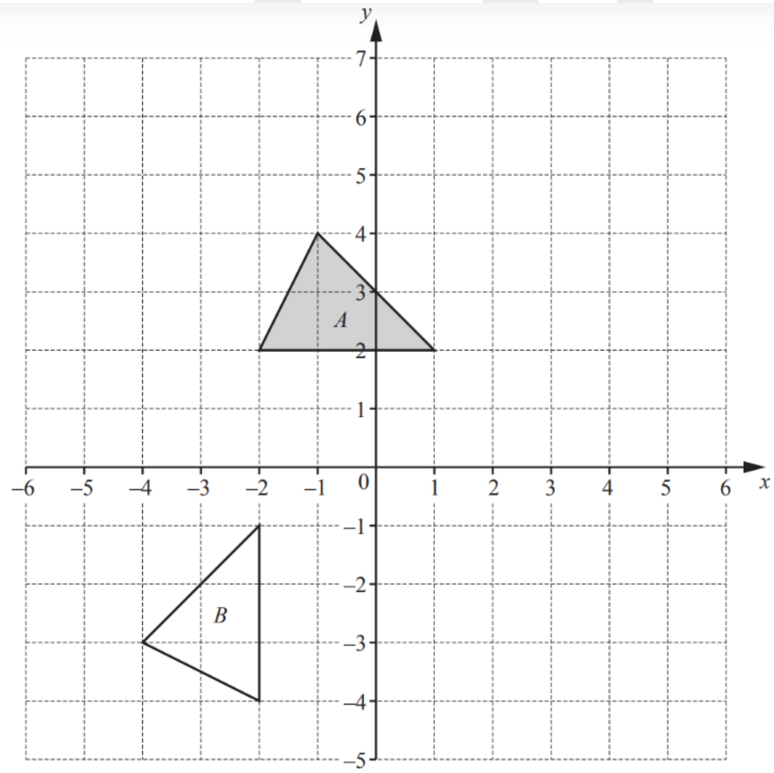
On the grid, draw the image of triangle *A* after a translation by the vector $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$.

10.



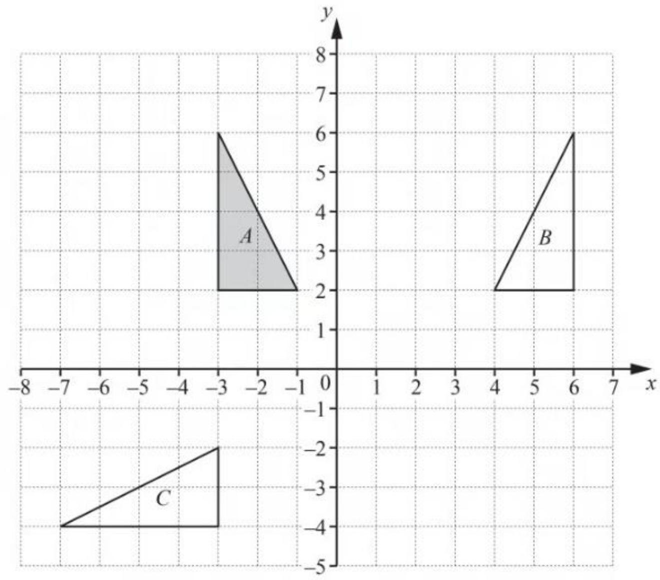
Describe fully the **single** transformation that maps triangle T onto triangle A .

11.



Describe fully the **single** transformation that maps triangle A onto triangle B .

12.

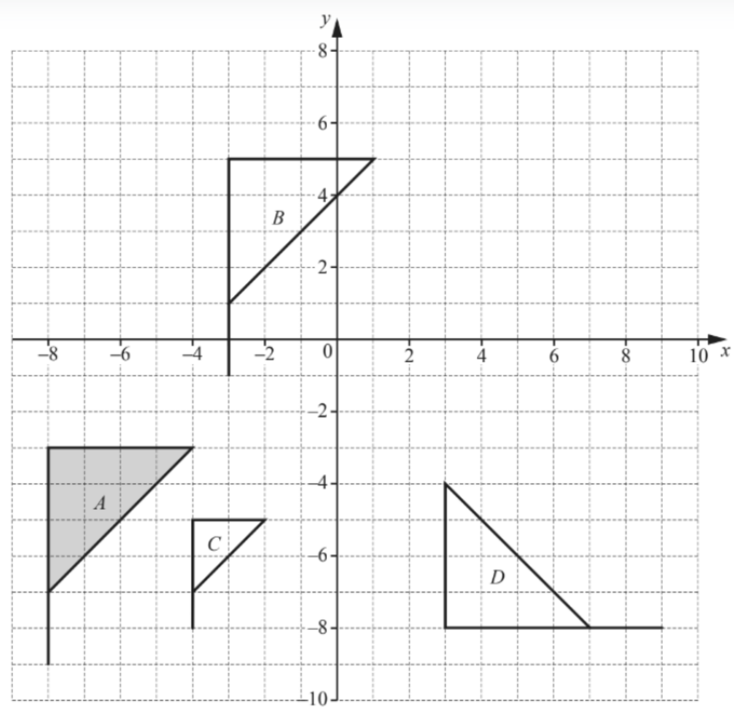


Describe fully the **single** transformation that maps

i) triangle *A* onto triangle *B*,

ii) triangle *A* onto triangle *C*.

13.



Describe fully the **single** transformation that maps flag *A* onto flag *D*.