Cambridge OL Mathematics

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Chapter 07 and Chapter 08

Indices and standard form

List of Indice Laws

- $x^0 = 1$
- $x^{-n} = \frac{1}{x^n}$
- $x^n \cdot x^m = x^{n+m}$
- $x^n \div x^{\mathrm{m}} = x^{n-m}$
- $(x^n)^m = x^{n \cdot \mathrm{m}}$
- $x^{\frac{n}{m}} = \sqrt[m]{x^n}$

Simplifying numbers with indices

- Indices (or powers) are a form of mathematical shorthand.
- You know that 4^2 means 4 × 4.
- The notation 4^2 is known as **index notation.**
- 2 is the **index** and 4 is the base
- Index notation can be used as shorthand for any power of a number.
- 3 × 3 × 3 × 3 can be written as 34 using index notation.
- 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 can be written as 28 using index notation.

Example 7.1

Question

Write each of these using index notation.

a $5 \times 5 \times 5 \times 5$

b $6 \times 6 \times 6 \times 4 \times 6 \times 4 \times 6 \times 4$

Solution

a $5 \times 5 \times 5 \times 5 = 5^4$

5 is multiplied 4 times, so the index is 4 and the base is 5.

b $6 \times 6 \times 6 \times 4 \times 6 \times 4 \times 6 \times 4 = 6^5 \times 4^3$

6 is multiplied 5 times so the index is 5 and the base is 6.

4 is multiplied 3 times so the index is 3 and the base is 4.

The two terms with different bases cannot be combined.

Note

The plural of 'index' is 'indices'.

Multiplying numbers in index form

When calculations involve more than one term with the same base, they can be simplified.

$3^4 \times 3^8 = (3 \times 3 \times 3 \times 3) \times (3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3)$

$= 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^{12}$

The indices are added.

$$3^4 \times 3^8 = 3^{4+8}$$
$$= 3^{12}$$

This gives a rule for multiplying numbers in index form.

$$n^a \times n^b = n^{a+b}$$

$$(3^4)^3 = 3^4 \times 3^4 \times 3^4$$
$$= 3^{4+4+4}$$
$$= 3^{3\times4}$$
$$= 3^{12}$$

This gives the rule

$$(n^a)^b = n^{a\times b}$$

Example 7.2

Question

Write these in a simpler form using indices.

a $4^3 \times 4^8$ **b** $9^2 \times 9 \times 9^4$ **c** $2^5 \times 3^2 \times 2^4$ **d** $(2^3)^2$

Solution

a $4^3 \times 4^8 = 4^{3+8} = 4^{11}$ The indices are added.

b $9^2 \times 9 \times 9^4 = 9^{2+1+4}$ Note that 9 is the same as 9^1.

$= 9^7$ The bases are all the same so the three indices are added.

c $2^5 \times 3^2 \times 2^4 = 2^{5+4} \times 3^2$ Only terms with the same base can be combined.

$= 2^9 \times 3^2$ There are two terms in the final answer.

d $(2^3)^2 = 2^{3\times2}$

$= 2^6$

Dividing numbers in index form

Divisions can be simplified by cancelling common factors.

$$2^6 \div 2^4 = \frac{\cancel{2} \times \cancel{2} \times \cancel{2} \times \cancel{2} \times 2 \times 2}{\cancel{2} \times \cancel{2} \times \cancel{2} \times \cancel{2}}$$
$$= 2 \times 2$$
$$= 2^2$$

The indices are subtracted.

$$2^6 \div 2^4 = 2^{6-4}$$
$$= 2^2$$

This gives a rule for dividing numbers in index form.

$$n^a \div n^b = n^{a-b}$$

Example 7.3

Question

Write these in a simpler form using indices.

a $8^6 \div 8^2$ **b** $\frac{7^9}{7 \times 7^3}$

Solution

a $8^6 \div 8^2 = 8^{6-2} = 8^4$ The indices are subtracted.

b $\frac{7^9}{7 \times 7^3} = \frac{7^9}{7^4}$ First simplify the denominator by adding the indices.

$= 7^{9-4}$ Then subtract the indices.

$= 7^5$

Negative indices

The index laws can also be used for numbers with negative indices.
Using the laws of indices

$5^0 \div 5^3 = 5^{0-3} = 5^{-3}$.

You also know that $5^0 = 1$, so

$5^0 \div 5^3 = 1 \div 5^3 = \frac{1}{5^3}$.

It follows that $5^{-3} = \frac{1}{5^3}$.

This gives a general law:

$$a^{-n} = \frac{1}{a^n}$$

Example 7.4

Question

Write these as fractions.

a 6^{-2} **b** 16^{-1} **c** n^{-3} **d** $\left(\frac{1}{4}\right)^{-2}$

Solution

a $6^{-2} = \frac{1}{6^2} = \frac{1}{36}$

b $16^{-1} = \frac{1}{16^1} = \frac{1}{16}$

c $n^{-3} = \frac{1}{n^3}$

d $\left(\frac{1}{4}\right)^{-2} = 4^2 = 16$

Fractional indices

Is there an index for a square or cube root?

Suppose that $a^b = \sqrt[3]{a}$.

Then $(a^b)^3 = \left(\sqrt[3]{a}\right)^3$ Cube both sides.

The cube of the cube root of a is equal to a, so the right-hand side of the equation is equivalent to a.

$a^{3b} = a = a^1$ Use the index law for powers on the left-hand side of the equation.

So $3b = 1$ Equate the powers.

$b = \frac{1}{3}$

So $a^{\frac{1}{3}} = \sqrt[3]{a}$.

This can be extended to give a general law:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

The index laws can be combined to work with more complex fractional powers.

For example, $a^{\frac{3}{2}} = \left(a^{\frac{1}{2}}\right)^3 = \left(\sqrt{a}\right)^3$

Also $a^{\frac{3}{2}} = (a^3)^{\frac{1}{2}} = \sqrt{a^3}$

A scientific calculator has a power key. It may look like this $x^{\blacksquare}$.

Find out how to use this key on your calculator to evaluate fractional and negative indices.

Key points

- Index notation is a mathematical shorthand with a base number and index (power).
- To multiply numbers with the same base, add the indices.
- To divide numbers with the same base, subtract the indices.
- The reciprocal of a number has a negative index.
- Any number with index zero is equal to 1.
- Fractional indices represent roots.

Example 7.5

Question

a Write $\sqrt[5]{n}$ using index notation. **b** Evaluate $16^{\frac{1}{4}}$.

c Evaluate $25^{-\frac{1}{2}}$. **d** Evaluate $125^{\frac{2}{3}}$.

Solution

a $\sqrt[5]{n} = n^{\frac{1}{5}}$

b $16^{\frac{1}{4}} = 2$ because $2^4 = 16$ so $16^{\frac{1}{4}} = (2^4)^{\frac{1}{4}} = 2^1 = 2$ Check this result on your calculator.

c $25^{-\frac{1}{2}} = \left(25^{\frac{1}{2}}\right)^{-1} = 5^{-1} = \frac{1}{5}$ Check this result on your calculator.

d $125^{\frac{2}{3}} = \left(\sqrt[3]{125}\right)^2 = 5^2 = 25$ Check this result on your calculator.

Note

Either the cube root or the square can be calculated first, the final result will be the same. It is often easier to work out the root first.

Chapter 08 - Standard form

Standard form is a way of making very large numbers and very small numbers easy to deal with.

In standard form, numbers are written as a number between 1 and 10 multiplied by a power of 10,

Large numbers

Example 8.1

Write these numbers in standard form.

Question	Solution
a 500 000	**a** $500\,000 = 5 \times 100\,000 = 5 \times 10^5$
b 6 300 000	**b** $6\,300\,000 = 6.3 \times 1\,000\,000 = 6.3 \times 10^6$
c 45 600	**c** $45\,600 = 4.56 \times 10\,000 = 4.56 \times 10^4$

Note

You can write down the answer without any intermediate steps.

Move the decimal point until the number is between 1 and 10.

Count the number of places the point has moved: that is the power of 10.

Small numbers

Example 8.2

Question

Write these numbers in standard form.

a 0.000 003 **b** 0.000 056 **c** 0.000 726

Note

You can write down the answer without any intermediate steps.

Move the decimal point until the number is between 1 and 10.

Count the number of places the point has moved, put a minus sign in front and that is the power of 10.

Solution

a $0.000\,003 = \frac{3}{1\,000\,000} = 3 \times \frac{1}{1\,000\,000} = 3 \times \frac{1}{10^6} = 3 \times 10^{-6}$

b $0.000\,056 = \frac{5.6}{100\,000} = 5.6 \times \frac{1}{100\,000} = 5.6 \times \frac{1}{10^5} = 5.6 \times 10^{-5}$

c $0.000\,726 = \frac{7.26}{10\,000} = 7.26 \times \frac{1}{10\,000} = 7.26 \times \frac{1}{10^4} = 7.26 \times 10^{-4}$

Key points

- A number written in standard form looks like $A \times 10^n$, where $1 \leqslant A < 10$ and n is a positive or negative integer.
- Without a calculator, know how to change numbers of any size into standard form and how to change standard form numbers into ordinary numbers.
- When adding or subtracting numbers in standard form without a calculator, change each to an ordinary number, perform the calculation and change the answer back to standard form.
- When multiplying or dividing numbers in standard form without a calculator, combine the numbers and the powers separately. Then, if necessary, change the answer back to standard form.
- Know how to input and read standard form numbers on a calculator.

Calculating with numbers in standard form

When you need to multiply or divide numbers in standard form, you can use your knowledge of the laws of indices.

Example 8.3

Question

Work out these. Give your answers in standard form.

a $(7 \times 10^3) \times (4 \times 10^4)$

b $(3 \times 10^8) \div (5 \times 10^3)$

Solution

a $(7 \times 10^3) \times (4 \times 10^4) = 7 \times 4 \times 10^3 \times 10^4$
$= 28 \times 10^7$
$= 2.8 \times 10^8$

b $(3 \times 10^8) \div (5 \times 10^3) = \frac{3 \times 10^8}{5 \times 10^3}$
$= 0.6 \times 10^5$
$= 6 \times 10^4$

Note

$10^3 \times 10^4 = 1000 \times 10\,000$
$= 10\,000\,000$
$= 10^7$

Note

$\frac{3}{5} = 0.6$ and

$\frac{10^8}{10^3} = \frac{100\,000\cancel{000}}{1\cancel{000}}$
$= 100\,000$
$= 10^5$

When you need to add or subtract numbers in standard form it is much safer to change to ordinary numbers first.

Example 8.4

Question

Work out these. Give your answers in standard form.

a $(7 \times 10^3) + (1.4 \times 10^4)$ **b** $(7.2 \times 10^5) + (2.5 \times 10^4)$ **c** $(5.3 \times 10^{-3}) - (4.9 \times 10^{-4})$

Solution

a
$$\begin{array}{r} 7\,000 \\ +\ 14\,000 \\ \hline 21\,000 \end{array} = 2.1 \times 10^4$$

b
$$\begin{array}{r} 720\,000 \\ +\ 25\,000 \\ \hline 745\,000 \end{array} = 7.45 \times 10^5$$

c
$$\begin{array}{r} 0.005\,30 \\ -\ 0.000\,49 \\ \hline 0.004\,81 \end{array} = 4.81 \times 10^{-3}$$

Revision questions

(a) It is given that p=4x10 and q=8x10° Expressing your answers in standard form, find

(i) $\frac{p}{q}$,

(ii) $\sqrt[3]{q}$.

(b) The numbers 225 and 540, written as the products of their prime factors, are
$225=3^2 x 5^2$,
$540=2^2 x 3^3 x 5$.

(i) Write 2250 as the product of its prime factors.
(ii) Find the smallest positive integer value of n for which 225n is a multiple of 540.

(2) By writing each number correct to 1 significant figure, estimate the value of

$$\frac{8.62 \times 2.04^2}{0.285}$$

3)The Earth is 1.5×10^8 kilometres from the Sun.

(a) Mercury is 5.81×10^2 kilometres from the Sun. How much nearer is the Sun to Mercury than to the Earth?
Give your answer in standard form.
(b) A terametre is 10 metres.
Find the distance of the Earth from the Sun in terametres.

3) It is given that = 2.1x107 and n=3x101. Expressing your answers in standard form, find
(a) m÷n,
(b) n^2+m.

4) (a) Convert 0.8 kilometres into millimetres.
(b) Evaluate $(6.3 \times 10^6) \div (9 \times 10^2)$, giving your answer in standard form.

5) Tom estimated the population of five countries in 2020. The table below shows these estimates.

Country	Population
Australia	2.35×10^7
Brazil	1.95×10^9
China	1.4×10^9
Japan	1.36×10^8
United Kingdom	6.9×10^7

(a) Which country did he estimate would have a population about 20 times that of the United Kingdom?

b) How many more people did he estimate would be in Japan than in Australia?
Give your answer in Standard form.

6)

$p = 8 \times 10^{-6}$ $q = 2 \times 10^{11}$

Evaluate the following, giving your answers in standard form.

(a) $p \times q$ [1]

(b) $p \div q$ [1]

(c) $\sqrt[3]{p}$ [1]

7)

$a^x = 5$

(a) Find a^{2x}.

(b) Find a^{-x}.

8)

(a) Evaluate $3^2 + 3^1 + 3^0$.

(b) Evaluate $\left(\frac{4}{3}\right)^{-2}$.

(c) Simplify $(16y^6)^{\frac{1}{2}}$.

9)

The table shows the populations, correct to 2 significant figures, of some African countries in 2014.

Country	Population
Nigeria	
Sudan	3.6×10^7
Chad	1.1×10^7
Namibia	2.2×10^6

(a) In 2014, the population of Nigeria was 177 156 000.
Complete the table with the population of Nigeria using standard form, correct to 2 significant figures. [2]

(b) Complete the following.
The population of Chad was .... times the population of Namibia. [1]

(c) The population density of a country is measured as the number of people per square kilometre.
It can be found by dividing the population of the country by its area in km².
The area of Sudan is 1.86×10^6 square kilometres.
Estimate the population density of Sudan.
Give your answer correct to 1 significant figure. [2]

10) The table shows information about the annual coffee production of some countries in 2010.

Country	Number of bags per year
Brazil	
Vietnam	1.85×10^7
Colombia	9.2×10^6
Indonesia	8.5×10^6

(a) In 2010, Brazil produced 48 million bags of coffee. Complete the table with the coffee production for Brazil, using standard form.

(b) How many more bags of coffee were produced in Vietnam than in Colombia?

(c) The mass of a bag of coffee is 60 kg.

Work out the number of kilograms of coffee produced in Indonesia.

Give your answer in standard form.

11) (a) Write the number 0.040 589 correct to 3 significant figures. [1]

(b) Giving your answer in standard form, evaluate $6\times10^{-4}+8\times10^{-5}$. [1]

(c) Estimate, correct to the nearest whole number, the value of $\sqrt{97}-\sqrt{35}$.

Show clearly the approximate values you use. [1]

12) The table below shows the populations of some countries in 2010.

Country	Population
Indonesia	2.4×10^8
Mexico	
Russia	1.4×10^8
Senegal	1.4×10^7
South Korea	4.8×10^7

(a) The population of Mexico was 111 210 000.
In the table above, complete the row for Mexico.
Give your answer in standard form, correct to two significant figures. [1]

(b) Complete the following sentences.

The population of Russia is ten times the population of ....

The population of .... is one fifth of the population of Indonesia. [2]

(c) Calculate the difference in population between South Korea and Senegal.
Give your answer in standard form. [1]