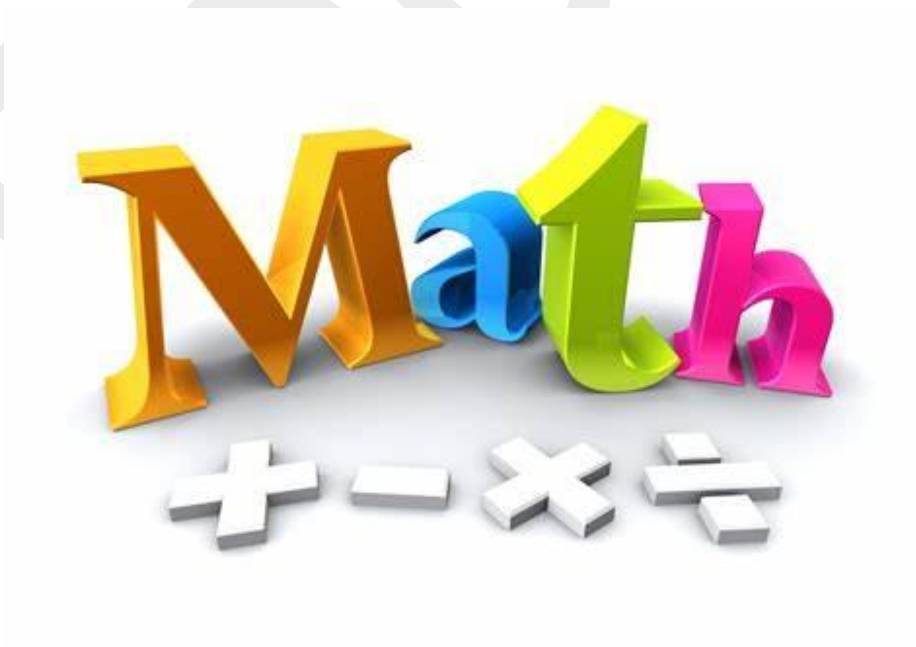
Cambridge OL

Mathematics

CODE: (4024)

Chapter 23

Equations



Writing formulas

Example 23.1

Question

Write a formula for the mean height, M cm, of a group of n people whose heights total h cm.

Solution

To find the mean height, you divide the total of the heights by the number of people, so the formula for M is

$M = h \div n$ or $M = \frac{h}{n}$

Note

If you are not sure whether to multiply or divide, try an example with numbers first.

Example 23.2

Question

The cost (\$$C$) of hiring a car is a fixed charge (\$$f$), plus the number of days (n) multiplied by the daily rate (\$$d$).

Write a formula for C.

Solution

$C = f + n \times d$ or $C = f + nd$

Writing equations

Some everyday problems can be solved by writing equations and solving them.

Example 23.3

Question

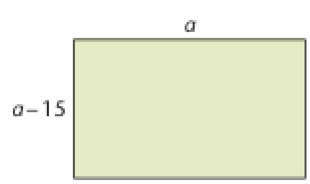
The length of a rectangle is a cm.

The width is 15 cm shorter.

The length is three times the width.

Write down an equation in a and solve it to find the length and width of the rectangle.

Solution



If the length = a, the width = $a - 15$.

Also the length = 3 × width = $3(a - 15)$.

The equation is $a = 3(a - 15)$.

$a = 3(a - 15)$	
$a = 3a - 45$	Multiply out the bracket.
$a + 45 = 3a$	Add 45 to each side.
$45 = 2a$	Subtract a from each side.
$22.5 = a$	Divide each side by 2.
$a = 22.5$	Rewrite with the subject on the left-hand side.

So the length = 22.5 cm and the width = 7.5 cm.

Solving

simple linear equations When you solve an equation, you always do the same thing to both sides of the equation. The following example will give you a reminder about solving simple linear equations.

Example 23.4

Question

Solve $5x + 2 = 12$.

Solution

$5x + 2 = 12$

$5x = 10$ Subtract 2 from both sides.

$x = 2$ Divide both sides by 5.

Solving equations with a bracket

Sometimes the equation formed to solve a problem will involve a bracket.

Example 23.5

Question

Solve $4(x - 5) = 18$.

Solution

By expanding the bracket, you will have an equation like that shown in Example 23.4.

Method 1

$4(x - 5) = 18$

$4x - 20 = 18$ Expand the brackets.

$4x = 38$ Add 20 to both sides.

$x = 9\frac{1}{2}$ Divide both sides by 4.

Note

You may find it easier to use Method 1.

Method 2

$4(x - 5) = 18$

$x - 5 = 4\frac{1}{2}$ Divide both sides by 4.

$x = 9\frac{1}{2}$ Add 5 to both sides.

Note

Method 2 is usually shorter.

Solving equations with the unknown on both sides

Sometimes the equation formed to solve a problem will have the unknown on both sides.

Example 23.6

Question

Solve $2(3x - 1) = 3(x - 2)$.

Solution

$2(3x - 1) = 3(x - 2)$

$6x - 2 = 3x - 6$ Expand the brackets.

$6x = 3x - 4$ Add 2 to both sides.

$3x = -4$ Subtract $3x$ from both sides.

$x = -1\frac{1}{3}$ Divide both sides by 3.

Solving equations involving fractions

You may have to solve equations involving fractions. The first step is to eliminate the fraction or fractions.

Example 23.7

Question

Solve $\frac{x}{3} = 2x - 3$.

Solution

$\frac{x}{3} = 2x - 3$

$x = 3(2x - 3)$ Multiply both sides by 3.

$x = 6x - 9$ Expand the bracket.

$x + 9 = 6x$ Add 9 to both sides.

$9 = 5x$ Subtract x from both sides.

Collecting the x-terms on the right-hand side of the equation makes the final x-term positive.

$x = \frac{9}{5}$ or $1\frac{4}{5}$ or 1.8 Divide both sides by 5.

Note
A common error when multiplying through by a number or letter is to multiply just the first term. Use a bracket to make sure.

Note
It is much easier to collect all the x-terms on to the side where the final x-term will be positive.

Note
Another common error in examples like this would be to give the answer as $\frac{5}{9}$ rather than $\frac{9}{5}$.

Example 23.8

Question

Solve the equation $\frac{400}{x} = 8$.

Solution

$\frac{400}{x} = 8$

$400 = 8x$ Multiply both sides by x.

$x = 50$ Divide both sides by 8.

Example 23.9

Question

Solve $\frac{x}{3} + \frac{x}{2} = 5$.

Solution

$\frac{x}{3} + \frac{x}{2} = 5$

$\frac{6x}{3} + \frac{6x}{2} = 5 \times 6$ Multiply every term by 6, because 6 is the common denominator of $\frac{x}{3}$ and $\frac{x}{2}$.

$\frac{{}^{2}\cancel{6}x}{\cancel{3}_1} + \frac{{}^{3}\cancel{6}x}{\cancel{2}_1} = 30$ Cancel the common factor in the numerator and denominator of each fraction.

$2x + 3x = 30$ This will eliminate both fractions.

$5x = 30$

$x = 6$ Divide both sides by 5.

Note
It is much easier to multiply both sides of the equation by a number that will eliminate the fraction than to expand the bracket.

Solving harder equations involving fractions

The following example illustrates how to solve harder equations involving fractions.

Example 23.10

Question

Solve $\frac{x-2}{4} - \frac{x+3}{5} = 1$.

Solution

$\frac{x-2}{4} - \frac{x+3}{5} = 1$ The common denominator of 4 and 5 is 20.

$\frac{20(x-2)}{4} - \frac{20(x+3)}{5} = 20$ Multiply every term by the common denominator.

$\frac{{}^{5}\cancel{20}(x-2)}{\cancel{4}_1} - \frac{{}^{4}\cancel{20}(x+3)}{\cancel{5}_1} = 20$ Cancel the common factors in the numerator and denominator of each fraction.

$5(x-2) - 4(x+3) = 20$ This will eliminate the fractions.

$5x - 10 - 4x - 12 = 20$ Expand the bracket. Take care with the signs.

$x - 22 = 20$ Collect like terms.

$x = 42$ Add 22 to both sides.

Note
The most common error is in the signs when the left-hand side is a subtraction. Make sure you put the brackets in.

Solving linear simultaneous equations

An equation in two unknowns does not have a unique solution.

When you are given two equations in two unknowns, such as x and y, they usually have a common solution where the two lines meet at a point. These are called simultaneous equations.

It is possible to solve simultaneous equations graphically. However, it can be time-consuming and you do not always obtain an accurate solution.

So, using algebra to solve them accurately is often better.

Example 23.11

Question

Solve the simultaneous equations $2x + 5y = 9$ and $2x - y = 3$.

Solution

$2x + 5y = 9$ (1) Write the two equations, one under the other, and label them.

$2x - y = 3$ (2)

This time (+)$2x$ appears in each equation, so subtract to eliminate the x-terms.

$2x - 2x + 5y - (-y) = 9 - 3$ (1) − (2). Take care with the signs.

$6y = 6$ $5y - (-y) = 5y + y$.

$y = 1$

$2x + 5 = 9$ Substitute $y = 1$ in (1). $5y$ is replaced by $5 \times 1 = 5$.

$2x = 4$

$x = 2$

The solution is $x = 2$, $y = 1$.

Check in equation (2): the left-hand side is $2x - y = 4 - 1 = 3$ which is correct.

Note

When eliminating, if the signs of the letter to be eliminated are the same, subtract. If they are different, add.

Note

When subtracting, take great care with the signs. If your check is wrong, see if you have made an error with any signs.

Example 23.12

Question

Solve the simultaneous equations $x + 3y = 10$ and $3x + 2y = 16$.

Solution

$x + 3y = 10$ (1) Write the two equations, one under the other, and label them.

$3x + 2y = 16$ (2)

The coefficients of x and y are different in the two equations.

Multiply (1) by 3 to make the coefficient of x the same as in equation (2).

$3x + 9y = 30$ (3) (1) × 3

$3x + 2y = 16$ (2)

Now (+)$3x$ appears in both equations, so subtract.

$3x - 3x + 9y - 2y = 30 - 16$ (3) − (2)

$7y = 14$

$y = 2$

$x + 6 = 10$ Substitute $y = 2$ in (1).

$x = 4$

The solution is $x = 4$, $y = 2$.

Check in equation (2): the left-hand side is $3x + 2y = 12 + 4 = 16$ which is correct.

Note

When subtracting equations, you can do equation (1) − equation (2) or equation (2) − equation (1). It is better to make the letter positive. Always write down clearly what you are doing.

Solving harder simultaneous equations

Sometimes the letters in the equations are not in the same order, so the first thing to do is to rearrange them. Sometimes each of the equations needs to be multiplied by a different number.

Example 23.13

Question

Solve simultaneously the equations $3y = 4 - 4x$ and $6x + 2y = 11$.

Solution

$4x + 3y = 4$ (1) Rearrange the first equation.

$6x + 2y = 11$ (2)

To eliminate x, multiply (1) by 3 and (2) by 2 and subtract,
or, to eliminate y, multiply (1) by 2 and (2) by 3 and subtract.

$12x + 9y = 12$ (3) (1) × 3

$12x + 4y = 22$ (4) (2) × 2

$5y = -10$ (3) − (4) Eliminate x.

$y = -2$

$4x - 6 = 4$ Substitute in (1). $3y$ is replaced by -6.

$4x = 10$

$x = \frac{10}{4} = \frac{5}{2} = 2\frac{1}{2}$

The solution is $x = 2\frac{1}{2}$, $y = -2$.

Check in equation (2): LHS = $6x + 2y = 15 - 4 = 11$, which is correct.

Note

If the equations are not already in the form $ax + by = c$, rearrange them so that they are.

Solving simultaneous equations using substitution

If x or y is the subject of one of the equations, there is an easier method to eliminate one of them. This method is called substitution and is shown in Example 23.14.

Example 23.14

Question

Solve simultaneously the equations $3x - 2y = 6$ and $y = 2x - 5$.

Solution

Since y is the subject of the second equation, we can substitute it in the first.

$3x - 2y = 6$ (1)

$y = 2x - 5$ (2)

$3x - 2(2x - 5) = 6$ Substitute (2) in (1). Replace y in equation (1) by $2x - 5$.

$3x - 2(2x - 5) = 6$ Solve the resulting equation.

$3x - 4x + 10 = 6$ Expand the bracket.

$-x = -4$

$x = 4$

$y = 2 \times 4 - 5$ Now substitute $x = 4$ into equation (2).

$y = 3$

So the solution is $x = 4$, $y = 3$.

Note

Always substitute the answer you have found first into the equation with x or y as the subject to find the other value.

The most common error is to forget the brackets when making the substitution.

Solving quadratic equations of the form $x^2 + bx + c = 0$ by factorisation

For any two numbers, if $A \times B = 0$, then either $A = 0$ or $B = 0$.

If $(x - 3)(x - 2) = 0$ then either $(x - 3) = 0$ or $(x - 2) = 0$.

To solve a quadratic equation, factorise it into two brackets and then use this fact.

Remember, to factorise $x^2 + bx + c$:

- if c is positive, you find two numbers with the same sign as b that multiply to give c and add to give b
- if c is negative, you find two numbers with different signs that multiply to give c and add to give b.

If an equation is written as $x^2 + bx = c$ or $x^2 = bx + c$, first rearrange it so that all the terms are on one side, leaving zero on the other side.

Example 23.15

Question

Solve the equation $x^2 = 4x - 4$ by factorisation.

Solution

$x^2 - 4x + 4 = 0$ Rearrange so that all the terms are on the same side.

$(x - 2)(x - 2) = 0$ Factorising: c is positive and b is negative, $-2 \times -2 = 4$, $-2 + -2 = -4$.

$x - 2 = 0$ or $x - 2 = 0$

The solution is $x = 2$ (repeated).

Note

There are always two solutions, so if they are both the same, write 'repeated'.

Example 23.16

Question

Solve the equation $x^2 + 5x = 0$.

Solution

$x(x + 5) = 0$ Factorising with x as a common factor.

$x = 0$ or $x + 5 = 0$

The solution is $x = 0$ or $x = -5$.

Example 23.17

Question

Solve the equation $x^2 - 49 = 0$.

Solution

$(x + 7)(x - 7) = 0$ Factorising by 'difference of two squares'.

$x + 7 = 0$ or $x - 7 = 0$

The solution is $x = -7$ or $x = 7$. This can be written as $x = \pm 7$.

You can use an alternative method for equations like this.

$x^2 - 49 = 0$

$x^2 = 49$ Add 49 to both sides.

$x = \pm 7$ Take the square root of both sides, remembering that this can give 7 or -7.

This method is perhaps simpler, but it is easy to forget the negative solution.

Solving quadratic equations of the form $ax^2 + bx + c = 0$ by factorisation

The following example has a question where the coefficient of x^2 is not 1.

Example 23.18

Question

Solve the equation $3x^2 + 11x + 6 = 0$.

Solution

$3x^2 + 11x + 6 = 0$

$(3x + 2)(x + 3) = 0$

$(3x + 2) = 0$ or $(x + 3) = 0$

$3x = -2$ or $x = -3$

$x = -\frac{2}{3}$ or $x = -3$

Solving quadratic equations by completing the square

This quadratic equation factorises.

$$x^2 - 4x + 3 = 0$$

This one does not.

$$x^2 - 4x + 1 = 0$$

$x^2 - 4x$ is part of the expansion $(x - 2)^2 = x^2 - 4x + 4$.

In this method for solving a quadratic equation, you 'complete the square' so that the left-hand side of the equation is a square $(mx + k)^2$.

When the coefficient of x^2 is 1,

$(x + k)^2 = x^2 + 2kx + k^2$,

so to find the number to add to complete the square, you halve the coefficient of x and square it.

Example 23.19

Question

Solve $x^2 - 4x + 1 = 0$.
Give your answer correct to 2 decimal places.

Solution

$x^2 - 4x + 1 = 0$

$x^2 - 4x + 1 + 3 = 0 + 3$ Add a number so the left-hand side is a complete square.

$x^2 - 4x + 4 = 3$

$(x - 2)^2 = 3$ Factorise the left-hand side.

$x - 2 = \sqrt{3}$ or $-\sqrt{3}$ Take the square root of both sides.

$x = 2 + \sqrt{3}$ or $2 - \sqrt{3}$ This is usually written as $x = 2 \pm \sqrt{3}$.

$x = 3.73$ or 0.27 to 2 decimal places

Note

Look carefully at the question. If it asks for an exact answer then you must give $2 + \sqrt{3}$ and $2 - \sqrt{3}$. If it asks for the answer to a given number of decimal places or significant figures, then you must give the decimal correctly rounded.

Example 23.20

Question

Write these quadratic expressions in completed square form, $a(x + b)^2 + c$.

a $x^2 + 6x - 3$

b $3x^2 + 15x + 14$

Solution

a $x^2 + 6x - 3 = (x + 3)^2 - 9 - 3$ — $x^2 + 6x$ is part of the expansion of $(x + 3)^2$, but without the +9 term, so subtract 9.

$= (x + 3)^2 - 12$

b $3x^2 + 15x + 14$ — Take out a factor of 3 from the terms involving x.

$= 3(x^2 + 5x) + 14$

$= 3\left[\left(x+\frac{5}{2}\right)^2 - \frac{25}{4}\right] + 14$ — Recognise that $(x^2 + 5x)$ is part of the expansion of $\left(x+\frac{5}{2}\right)^2$ but without the term $\left(\frac{5}{2}\right)^2$.

Remember that you need to multiply the $-\frac{25}{4}$ term by 3.

$= 3\left(x+\frac{5}{2}\right)^2 - \frac{75}{4} + 14$

$= 3\left(x+\frac{5}{2}\right)^2 - \frac{19}{4}$

Solving quadratic equations using the quadratic formula

Using the method of completing the square it may be shown that

the equation $ax^2 + bx + c = 0$ has roots $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example 23.21

Question

Solve the equation $3x^2 + 4x - 2 = 0$.

Give your answers to 2 decimal places.

Solution

In the equation $3x^2 + 4x - 2 = 0$, $a = 3$, $b = 4$, $c = -2$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-4 \pm \sqrt{16 - 4 \times 3 \times (-2)}}{2 \times 3}$$

$$= \frac{-4 \pm \sqrt{16 + 24}}{6}$$

$$= \frac{-4 \pm \sqrt{40}}{6}$$

$$= \frac{-4 + 6.324\ldots}{6}$$

$$= \frac{-4 + 6.324}{6} \text{ or } \frac{-4 - 6.324}{6}$$

$x = 0.39$ or -1.72 to 2 decimal places

Note

The main errors that occur in using the formula are

- errors with the signs, especially with $-4ac$
- failure to divide the whole expression by $2a$.

Note

You do not need to memorise the quadratic formula but you should be able to use it correctly.

Changing the subject of formulas

When solving equations, you simplify and collect terms if necessary, then use inverse operations to get the unknown on its own. Formulas can be treated in the same way as equations. This means they can be rearranged to change the subject. Use the same steps as you would if the formula was an equation.

Example 23.22

Question

Rearrange each of these formulas to make the letter in the bracket the subject.

a $a = b + c$ (b) **b** $a = bx + c$ (b) **c** $n = m - 3s$ (s) **d** $p = \frac{q+r}{s}$ (r)

Solution

a $a = b + c$

$a - c = b$ Subtract c from both sides.

$b = a - c$ Reverse to get b on the left.

b $a = bx + c$

$a - c = bx$ Subtract c from both sides.

$\frac{a-c}{x} = b$ Divide both sides by x.

$b = \frac{a-c}{x}$ Reverse to get b on the left.

c $n = m - 3s$

$n + 3s = m$ Add $3s$ to both sides.

$3s = m - n$ Subtract n from both sides.

$s = \frac{m-n}{3}$ Divide both sides by 3.

d $p = \frac{q+r}{s}$

$sp = q + r$ Multiply both sides by s.

$sp - q = r$ Subtract q from both sides.

$r = sp - q$ Reverse to get r on the left.

Note

If you are not sure whether to multiply or divide, try an example with numbers first.

Example 23.23

Question

The formula for the total cost, T, of entry to the cinema for three adults and three children is given by the formula $T = 3(a + c)$, where a is the price of an adult ticket and c is the price of a child ticket.

a Make c the subject of the formula.

b Find the cost for a child when the cost for an adult is \$5 and the total cost is \$24.

Solution

a $T = 3(a + c)$

$T = 3a + 3c$ Multiply out the bracket.

$T - 3a = 3c$ Subtract $3a$ from both sides.

$\frac{T-3a}{3} = c$ Divide both sides by 3.

$c = \frac{T-3a}{3}$ Reverse to get c on the left.

b $c = \frac{24 - 3 \times 5}{3}$

$= \frac{24-15}{3}$

$= \frac{9}{3} = 3$

The cost for a child is \$3.

Changing the subject of harder formulas

All the formulas that you have rearranged up to this point have contained the new subject only once, and also the subject has not been raised to a power. However, this is now extended in the following examples.

Example 23.24

Question

Rearrange the formula $V = \frac{4}{3}\pi r^3$ to make r the subject.

Solution

$V = \frac{4}{3}\pi r^3$	
$3V = 4\pi r^3$	Multiply both sides by 3.
$4\pi r^3 = 3V$	Rearrange to get all terms involving r on the left.
$r^3 = \frac{3V}{4\pi}$	Divide both sides by 4π.
$r = \sqrt[3]{\frac{3V}{4\pi}}$	Take the cube root of both sides.

Example 23.25

Question

Rearrange the formula $a = x + \frac{cx}{d}$ to make x the subject.

Solution

$a = x + \frac{cx}{d}$	
$ad = dx + cx$	Multiply both sides by d.
$dx + cx = ad$	Rearrange to get all terms involving x (the subject) on the left.
$x(d + c) = ad$	Factorise the left-hand side, taking out the factor x.
$x = \frac{ad}{d + c}$	Divide both sides by the bracket $(d + c)$.

Solving equations involving algebraic fractions

You need to be able to solve equations involving fractions with algebraic expressions in the denominator. You deal with these in the same way as you dealt with fractions with numerical denominators. You multiply every term by the common denominator to eliminate the fractions. The resulting equation may be linear or quadratic.

Example 23.28

Question

Solve $\frac{3}{x+1} - \frac{2}{x} = \frac{1}{x-2}$.

Solution

$\frac{3}{x+1} - \frac{2}{x} = \frac{1}{x-2}$	The common denominator of $x + 1$, x and $x - 2$ is $x(x + 1)(x - 2)$.
$3x(x - 2) - 2(x + 1)(x - 2) = x(x + 1)$	Multiply by $x(x + 1)(x - 2)$. Make sure you multiply every expression on both sides.
$3x^2 - 6x - 2(x^2 - x - 2) = x^2 + x$	Expand the bracket and collect on one side.
$3x^2 - 6x - 2x^2 + 2x + 4 - x^2 - x = 0$	
$-5x + 4 = 0$	Collect like terms.
$4 = 5x$	
$x = \frac{4}{5}$	

Note

In equations with algebraic fractions, multiply through by the common denominator. This eliminates the fractions.

Key points

- The two sides of an equation must always be kept equal. Operations used to simplify or solve an equation must always be the same for each side.
- To solve equations with x on both sides, collect the x terms together on one side of the equation.
- To solve equations with brackets, multiply out the brackets first.
- To solve equations with fractions, first multiply every term in the equation by the lowest common denominator of the fractions. Then, multiply out any brackets.
- To write a formula or equation, think first what operations you would use for the first line of a solution if the letters were numbers. Then, write those operations using the letters instead.
- Some problems can be solved by writing an equation and then solving it.
- To solve two linear simultaneous equations using algebra, you can use the elimination method or the substitution method.
- To solve a quadratic equation by factorisation, first rearrange the equation, if necessary, so that all the terms are on the left-hand side, with zero on the right-hand side. Next, factorise the left-hand side. Lastly, set each bracket to zero to find the two solutions.
- Quadratic equations which will not factorise may be solved by completing the square.
  - Rearrange the equation, if needed, so that all the terms with x or x^2 are on the left-hand side.
  - Multiply, if necessary, so that the coefficient of x^2 is a perfect square.
  - Add a number to the left-hand side so that the left-hand side is a complete square which will factorise to $(ax + b)^2$.
  - Take the square root of both sides.
  - Solve the two resulting linear equations.
- The roots of the quadratic equation $ax^2 + bx + c = 0$ may be found by substituting into this formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- When cancelling algebraic fractions, factorise if necessary. Only cancel factors.
- Rearrange a formula using the same steps as you would if it were an equation with numbers instead of letters.
- To rearrange a formula where the new subject occurs twice, first rearrange the formula so that all the terms containing the new subject are on one side and the remaining terms are on the other side. Then, take the new subject out as a common factor. Lastly, divide by the other factor.
- When a formula contains a power or root of a new subject, rearrange the formula to get that term by itself. Then, use the inverse operation. For example, if the formula has $\sqrt{a}$, then to find a, square the other side of the formula.

Revision questions

1. Write $x^2 - 4x + 7$ in the form $(x - a)^2 + b$.

2. Write down the coordinates of the turning point of the graph of $y = x^2 - 4x + 7$.

3. Write $x^2 + 10x + 14$ in the form $(x + a)^2 + b$.

4. $x^2 + 4x - 9 = (x + a)^2 + b$

   Find the value of a and the value of b.

5. i) Write $x^2 + 8x - 9$ in the form $(x + k)^2 + h$.

   ii) Use your answer to **part (i)** to solve the equation $x^2 + 8x - 9 = 0$.

6. Solve by factorisation $10r^2 - 23r + 9 = 0$.

7. $x^2 - 12x + a = (x + b)^2$

   Find the value of a and the value of b.

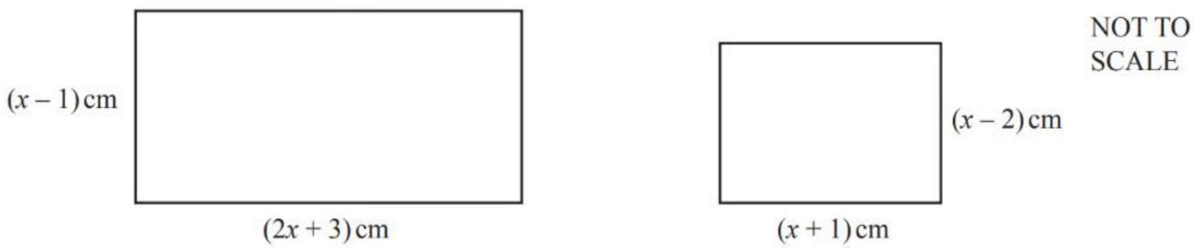
8. Write $x^2 - 6x - 19$ in the form $(x - a)^2 + b$.

9.

The solutions of the equation $x^2 + bx + c = 0$ are $\frac{-7+\sqrt{61}}{2}$ and $\frac{-7-\sqrt{61}}{2}$.

Find the value of b and the value of c.

10.



The difference between the areas of the two rectangles is 62 cm^2.

i) Show that $x^2 + 2x - 63 = 0$.

[3]

ii) Factorise $x^2 + 2x - 63$.

[2]

iii) Solve the equation $x^2 + 2x - 63 = 0$ to find the difference between the perimeters of the two rectangles.

.... cm [2]