Edexcel

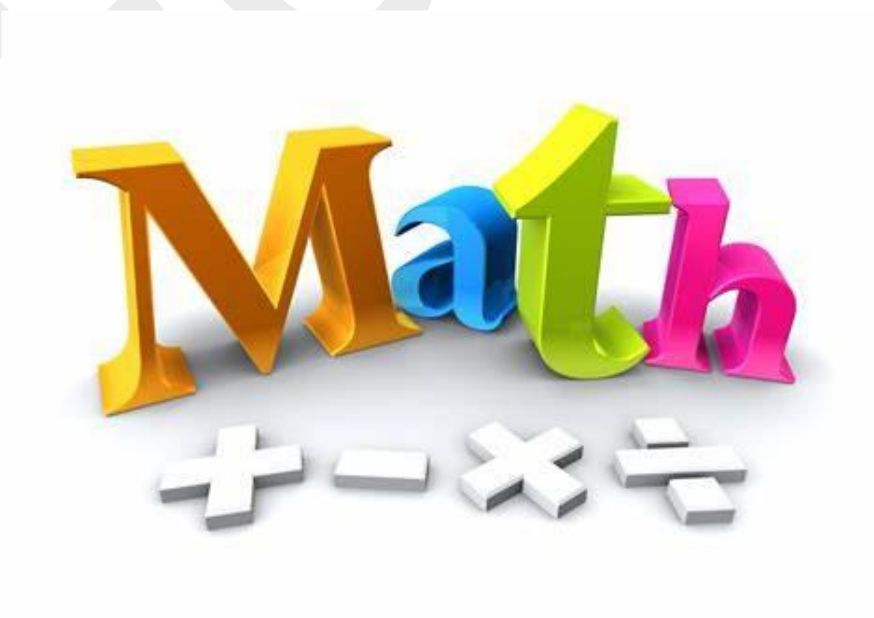
OL IGCSE

Mathematics

CODE: (4CP0)

Unit 09

4MB1



NUMBER 9

BASIC PRINCIPLES

- Global financial processes can be complex. The ones in this section involve the simple day-to-day concepts of comparative costs, salaries and taxes, sales tax and foreign currency.
- The mathematical processes involved in this section have all been met before.
- The key skills all involve percentages.
- To calculate x **as a percentage of** y: $\frac{x}{y} \times 100$
- To calculate x **per cent of** y: 1% of $y = \frac{y}{100}$

  so x% of $y = x \times \frac{y}{100} = y \times \left(\frac{x}{100}\right)$
- 5% of a quantity can be found by multiplying by $\frac{5}{100}$ or 0.05
- 95% of a quantity can be found by multiplying by $\frac{95}{100}$ or 0.95
- $1\% = \frac{1}{100} = 0.01$ $\quad 10\% = \frac{10}{100} = \frac{1}{10} = 0.1$
- $50\% = \frac{50}{100} = \frac{1}{2} = 0.5$ $\quad 75\% = \frac{75}{100} = \frac{3}{4} = 0.75$
- Percentage change $= \frac{\text{value of change}}{\text{original value}} \times 100$
- Per annum (p.a.) is frequently used and means 'per year'.

PERCENTAGE INCREASE AND DECREASE

To increase a quantity by R%, multiply it by $1 + \frac{R}{100}$

To decrease a quantity by R%, multiply it by $1 - \frac{R}{100}$

PERCENTAGE CHANGE	MULTIPLY BY
+ 5%	1.05
+ 95%	1.95
− 5%	0.95
− 95%	0.05

COMPARATIVE COSTS

To help shoppers to compare the prices of packaged items, shopkeepers often show the cost per 100 grams or cost per litre. Other units will enable a consumer to compare the value of items they might wish to buy.

EXAMPLE 1

SKILLS

PROBLEM SOLVING

A 100 g jar of Brazilian coffee costs €4.50, and a 250 g jar of the same coffee costs €12.

Which is the better value?

100 g jar:

cost/g $= \frac{450}{100}$ cents/g

$= 4.50$ cents/g

250 g jar:

cost/g $= \frac{1200}{250}$ cents/g

$= 4.80$ cents/g

The 100 g jar of coffee is better value.

TAXATION

Governments collect money from their citizens through tax to pay for public services such as education, health care, military defence and transport.

SALES TAX

This is the tax paid on spending. It is included in the price paid for most articles. In many countries some articles are free from sales tax, for example, children's clothing.

EXAMPLE 2

SKILLS PROBLEM SOLVING

Rita buys a tennis racket for \$54 plus sales tax at 15%.
Calculate the selling price.

Selling price = \$54 × 1.15 = \$62.10

EXAMPLE 3

SKILLS PROBLEM SOLVING

Liam buys a computer game for \$36 including a sales tax of 20%.
Calculate the sales tax.

Let the price excluding sales tax be \$$p$.

$p \times 1.20 = 36$

$p = \frac{36}{1.20} = 30$

Sales tax = \$36 − \$30 = \$6

SALARIES AND INCOME TAX

A salary is a fixed annual sum of money usually paid each month, but the salary is normally stated per annum (p.a.).

Income tax is paid on money earned and most governments believe that richer people should pay more tax than poorer people.

As a result of this the amount of tax falls into different rates or 'tax bands'.

The country of Kalculus has these tax rates.

TAX RATES	SALARY P.A
20%	\$0–\$50 000
40%	> \$50 000

The examples below explain how these tax rates are used to find the tax owed. Any salary over \$50 000 p.a. pays 40% on the amount over \$50 000 paid p.a.

EXAMPLE 4

SKILLS PROBLEM SOLVING

Lauren earns \$30 000 p.a. as a waitress in the country of Kalculus.
What is her monthly income tax bill?

Tax rate on \$30 000 is 20%.

Annual tax = \$30 000 × 0.20 = \$6000

Monthly tax bill = $\frac{\$6000}{12}$ = \$500

EXAMPLE 5

SKILLS

PROBLEM SOLVING

Frankie earns $1 000 000 p.a. as a professional footballer in the country of Kalculus. What is his monthly income tax bill?

The first $50 000 is taxed at 20%, the remainder is taxed at 40%.

Annual tax at 20%:	$50 000 × 0.20	= $10 000	p.a.
Annual tax at 40%:	($1 000 000 − $50 000) × 0.40	= $380 000	p.a.
Total tax p.a.	$10 000 + $380 000	= $390 000	p.a.

Monthly income tax bill = $\frac{\$390\,000}{12}$ = $32 500

FOREIGN CURRENCY

Different currencies are exchanged and converted around the world, so an agreed rate of conversion is needed. The table below shows some examples of values compared to $1 (USA)

COUNTRY OR CONTINENT	CURRENCY	EXCHANGE RATE TO $1 (US dollar)
Australia	dollar	$1.46
China	yuan	元6.57
Europe	euro	€0.92
India	rupee	₹67.79
Nigeria	naira	₦199.25
Russia	ruble	₽77.75
South Africa	rand	R16.78
UK	pound	£0.70

EXAMPLE 6

SKILLS

PROBLEM SOLVING

How many euros will $150 (US dollars) buy?

Using the conversion rates in the table above:

$1 = €0.92

$150 = 150 × €0.92 = €138

EXAMPLE 7

SKILLS

PROBLEM SOLVING

How many UK pounds will 元150 buy?

Using the conversion rates in the table on page 271:

$1 = 元6.57 (Divide both sides by 6.57)

$0.15 = 元1

元150 = 150 × $0.15 = $22.50

$22.50 = 22.50 × £0.70 = £15.75

ALGEBRA 9

BASIC PRINCIPLES

- Solve **quadratic equations** (using **factorisation** or the quadratic formula).
- Solve simultaneous equations (by substitution, elimination or graphically).
- **Expand** brackets.
- Expand the **product** of two linear expressions.
- Form and **simplify** expressions.
- Factorise expressions.
- Complete the square for a quadratic expression.

SOLVING TWO SIMULTANEOUS EQUATIONS – ONE LINEAR AND ONE NON-LINEAR

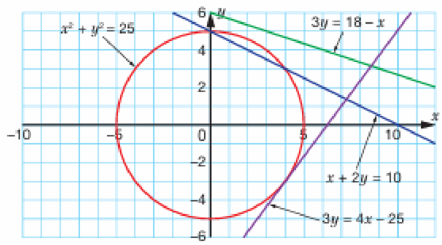
ACTIVITY 1

SKILLS
ANALYSIS

Use the graph to solve the simultaneous equations
$x + 2y = 10$ and $x^2 + y^2 = 25$

What is the connection between the line $3y = 4x - 25$ and the circle $x^2 + y^2 = 25$?

Are there any real solutions to the simultaneous equations
$3y = 18 - x$ and $x^2 + y^2 = 25$?



KEY POINTS

- When solving simultaneous equations where one equation is linear and the other is non-linear:
  - If there is one solution, the line is a **tangent** to the curve.
  - If there is no solution, the line does not **intersect** the curve.

Drawing graphs is one way of solving simultaneous equations where one equation is linear and the other quadratic. Sometimes they can be solved algebraically.

EXAMPLE 1

SKILLS
ANALYSIS

Solve the simultaneous equations $y = x + 6$ and $y = 2x^2$ algebraically and show the result graphically.

$y = x + 6$ (1)

$y = 2x^2$ (2)

Substitute (2) into (1):

$2x^2 = x + 6$ (Rearrange)

$2x^2 - x - 6 = 0$ (Factorise)

$(2x + 3)(x - 2) = 0$ (Solve)

So either $(2x + 3) = 0$ or $(x - 2) = 0$

$x = -1\frac{1}{2}$ or $x = 2$

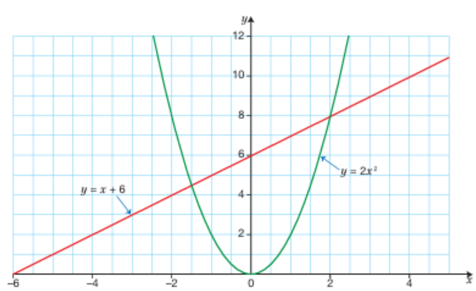
Substitute $x = -1\frac{1}{2}$ into (1) to give $y = 4\frac{1}{2}$

Substitute $x = 2$ into (1) to give $y = 8$

So the solutions are $x = -1\frac{1}{2}$, $y = 4\frac{1}{2}$ and $x = 2$, $y = 8$

The graphs of the equations are shown below.

The solutions correspond to the intersection points $(-1\frac{1}{2}, 4\frac{1}{2})$ and $(2, 8)$.



KEY POINTS

- If the two equations are of the form $y = f(x)$ and $y = g(x)$:
  - Solve the equation $f(x) = g(x)$ to find x.
  - When x has been found, find y using the easier of the original equations.
  - Write out your solutions in the correct pairs.

EXAMPLE 2

SKILLS

ANALYSIS

Solve the simultaneous equations $x + 2y = 10$ and $x^2 + y^2 = 25$

$x + 2y = 10$ (1)

$x^2 + y^2 = 25$ (2)

Make x the **subject** of equation (1) (the linear equation):

$x = 10 - 2y$ (3)

Substitute (3) into (2) (the non-linear equation):

$(10 - 2y)^2 + y^2 = 25$ (Expand brackets)

$100 - 40y + 4y^2 + y^2 = 25$ (Simplify)

$5y^2 - 40y + 75 = 0$ (Divide both sides by 5)

$y^2 - 8y + 15 = 0$ (Solve by factorising)

$(y - 3)(y - 5) = 0$

y = 3 or y = 5 Substitute y = 3 into (1) to give x = 4
Substitute y = 5 into (1) to give x = 0
So the solutions are x = 0, y = 5 and x = 4, y = 3

EXAMPLE 3

SKILLS

ANALYSIS

Solve the simultaneous equations $x + y = 4$ and $x^2 + 2xy = 2$

$x + y = 4$ (1)

$x^2 + 2xy = 2$ (2)

Substituting for y from (1) into (2) will make the working easier.

Make y the subject of (1):

$y = 4 - x$ (3)

Substitute (3) into (2):

$x^2 + 2x(4 - x) = 2$ (Expand brackets)

$x^2 + 8x - 2x^2 = 2$ (Simplify)

$x^2 - 8x + 2 = 0$ (Solve using the quadratic formula)

$$x = \frac{8 \pm \sqrt{(-8)^2 - 4 \times 1 \times 2}}{2 \times 1}$$

$x = 0.258$ or $x = 7.74$ (to 3 s.f.)

Substituting $x = 0.258$ into (1) gives $y = 3.74$ (to 3 s.f.)

Substituting $x = 7.74$ into (1) gives $y = -3.74$ (to 3 s.f.)

So the solutions are
$x = 0.258$ and $y = 3.74$ or $x = 7.74$ and $y = -3.74$ (to 3 s.f.)

PROOF

Beal's Conjecture There was once a billionaire called Beal, Pursued his Conjecture with zeal, If someone could crack it They'd make a right packet: But only genius will clinch'em the deal. Limerick by Rebecca Siddall (an IGCSE student)

There is as yet no proof of Beal's Conjecture, made by Andrew Beal in 1993, which is 'If ax + by = c z then a, b and c must have a common prime factor.' (All letters are positive integers with x, y and z > 2.) Andrew Beal himself has offered a reward of $106 for a proof or counter-example to his conjecture.

To prove that a statement is true you must show that it is true in all cases. However, to prove that a statement is not true all you need to do is find a counter-example.

A counter-example is an example that does not fit the statement

EXAMPLE 4

SKILLS ANALYSIS

Bailey says that substituting integers for n in the expression $n^2 + n + 1$ always produces **prime numbers**. Give a counter-example to prove that Bailey's statement is not true.

Substitute different integers for n.

When $n = 1$, $n^2 + n + 1 = 3$ which is prime.

When $n = 2$, $n^2 + n + 1 = 7$ which is prime.

When $n = 3$, $n^2 + n + 1 = 13$ which is prime.

When $n = 4$, $n^2 + n + 1 = 21$ which is not prime.

Disproving statements by considering a lot of cases and trying to find a counter-example can take a long time, and will not work if the statement turns out to be true. For example, the formula $2n^2 + 29$ will produce prime numbers up to $n = 28$ and most people would have given up long before the solution is found!

EXAMPLE 5

SKILLS ANALYSIS

Find a counter-example to prove that $2n^2 + 29$ is not always prime.

Substitute $n = 29$ because $2n^2 + 29$ will then factorise.

When $n = 29$, $2 \times 29^2 + 29 = 1711$

However $2 \times 29^2 + 29 = 29(2 \times 29 + 1) = 29 \times 59$

So $1711 = 29 \times 59$ showing that 1711 is not prime.

EXAMPLE 6

Prove that the difference between the squares of any two **consecutive** integers is equal to the sum of these integers.

Let n be any integer. Then the next integer is $n + 1$

The difference between the squares of these two consecutive integers is

$(n + 1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1$

The sum of these two consecutive integers is $n + (n + 1) = 2n + 1$

So the difference between the squares of any two consecutive integers is equal to the sum of these integers.

KEY POINT

- When n is an integer, consecutive integers can be written in the form $\ldots, n - 1, n, n + 1, n + 2, \ldots$

EVEN AND ODD NUMBER PROOFS

An even number is divisible by two, so an even number can always be written as 2 × another number or 2n where n is an integer. An odd number is one more than an even number so 2n + 1 is always odd where n is an integer

EXAMPLE 7

SKILLS

ANALYSIS

Prove that the product of any two odd numbers is always odd.

Let the two odd numbers be $2n + 1$ and $2m + 1$

Then $(2n + 1)(2m + 1) = 4nm + 2n + 2m + 1$

$= 2(2nm + n + m) + 1$

But $2nm + n + m$ is an integer. Let $2nm + n + m = p$

Then $(2n + 1)(2m + 1) = 2p + 1$ which is odd.

KEY POINTS

- When n is an integer
  - Any even number can be written in the form $2n$.
  - Consecutive even numbers can be written in the form $2n$, $2n + 2$, $2n + 4$, …
  - Any odd number can be written in the form $2n + 1$.
  - Consecutive odd numbers can be written in the form $2n + 1$, $2n + 3$, $2n + 5$, …

SKILLS

ANALYSIS

ACTIVITY 3

Here are two 'proofs'. Try to find the mistakes.

1 ▶ Let $a = b$

$\Rightarrow a^2 = ab$

$\Rightarrow a^2 - b^2 = ab - b^2$

$\Rightarrow (a + b)(a - b) = b(a - b)$

$\Rightarrow a + b = b$

Substituting $a = b = 1$ gives $2 = 1$!

2 ▶ Assume a and b are positive and that $a > b$.

$a > b$

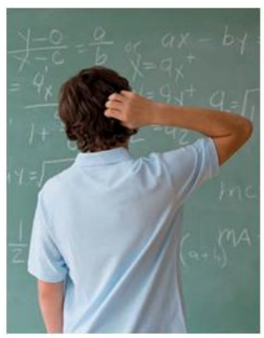
$\Rightarrow ab > b^2$

$\Rightarrow ab - a^2 > b^2 - a^2$

$\Rightarrow a(b - a) > (b + a)(b - a)$

$\Rightarrow a > b + a$

Substituting $a = 2, b = 1$ leads to $2 > 3$!



PROOFS USING COMPLETING THE SQUARE

EXAMPLE 8

SKILLS

ANALYSIS

Prove that $x^2 - 8x + 16 \geq 0$ for any value of x.
Hence **sketch** the graph of $y = x^2 - 8x + 16$

$x^2 - 8x + 16 = (x - 4)^2 - 16 + 16$ (by completing the square)

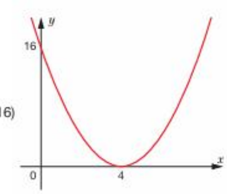
$= (x - 4)^2$

$(x - 4)^2 \geq 0$ for any value of x as any number squared is always positive, so $x^2 - 8x + 16 \geq 0$ for any value of x.

To sketch $y = (x - 4)^2$, note that when $x = 4$, $(x - 4)^2 = 0$ so the point $(4, 0)$ lies on the graph.

This point must be the **minimum point** on the graph because $(x - 4)^2 \geq 0$. When $x = 0$, $y = 16$ (by substituting in $y = x^2 - 8x + 16$) so $(0, 16)$ lies on the graph.

The graph is also a positive parabola so is 'U' shaped.



EXAMPLE 9

SKILLS ANALYSIS

Prove that $8x - x^2 - 18 < 0$ for any value of x.
Hence find the largest value of $8x - x^2 - 18$ and sketch the graph of $y = 8x - x^2 - 18$

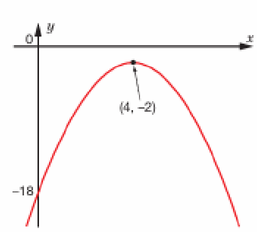
$8x - x^2 - 18 = -(x^2 - 8x + 18)$
$= -((x - 4)^2 - 16 + 18)$ (by completing the square)
$= -((x - 4)^2 + 2)$
$= -(x - 4)^2 - 2$

$(x - 4)^2 \geq 0$ for any value of $x \Rightarrow -(x - 4)^2 \leq 0$
so $-(x - 4)^2 - 2 < 0$ for any value of x.

When $x = 4$, $-(x - 4)^2 = 0$
so the largest value of $-(x - 4)^2 - 2$ is -2.

To sketch $y = -(x - 4)^2 - 2$, note that when $x = 4$, $y = -2$ and that this is the largest value of y so this is a **maximum point**.

Also when $x = 0$, $y = -18$ (substituting in $y = 8x - x^2 - 18$) so $(0, -18)$ lies on the graph. The graph is also a negative **parabola** so is '∩' shaped.



KEY POINTS

- $(x - a)^2 \geq 0$ and $(x + a)^2 \geq 0$ for all x.
- $(x - a)^2 = 0$ when $x = a$ and $(x + a)^2 = 0$ when $x = -a$.
- To prove a quadratic **function** is greater or less than zero, write it in completed square form.
- To find the co-ordinates of the **turning point** of a **quadratic graph**, write it in completed square form $y = a(x + b)^2 + c$. The turning point is then $(-b, c)$.

GRAPHS 8

BASIC PRINCIPLES

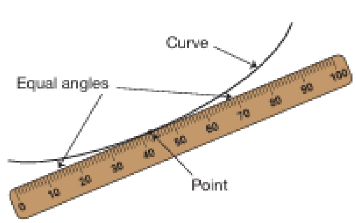
- Remember that a **tangent** is a straight line that touches a curve at one point only.
- Find the **gradient** of a line through two points.
- Find the gradient of a straight-line graph.
- Plot the graphs of linear and quadratic **functions** using a table of values.
- Interpret distance–time graphs.
- Interpret speed–time graphs.
- Identify transformations.
- Translate a shape using a **vector** and describe a **translation** using a **column vector**.
- Reflect a shape in the x- and y-axes and describe a reflection.
- Identify the image of a point after a reflection in the x-axis or a reflection in the y-axis.
- Use function notation.

GRADIENT OF A CURVE AT A POINT

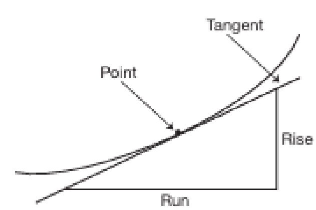
Most graphs of real-life situations are curves rather than straight lines, but information on rates of change can still be found by drawing a tangent to the curve and using this to estimate the gradient of the curve at that point.

To find the gradient of a curve at a point, draw the tangent to the curve at the point. Do this by pivoting a ruler about the point until the angles between the ruler and the curve are as equal as possible.

The gradient of the tangent is the gradient of the curve at the point.

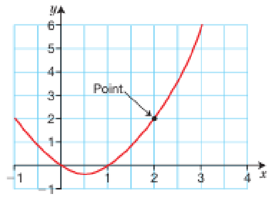


The gradient of the tangent is found by working out $\frac{\text{'rise'}}{\text{'run'}}$



EXAMPLE 1

The graph of $y = x^2 - x$ is shown. Find the gradient of the graph at $x = 2$



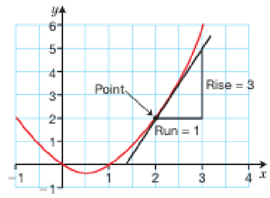
Use a ruler to draw a tangent to the curve at $x = 2$

Work out the rise and run.

Note: Be careful finding the rise and run when the **scales** on the axes are different as in this example.

The gradient of the tangent is $\frac{\text{rise}}{\text{run}} = \frac{3}{1} = 3$

So the gradient of the curve at $x = 2$ is 3



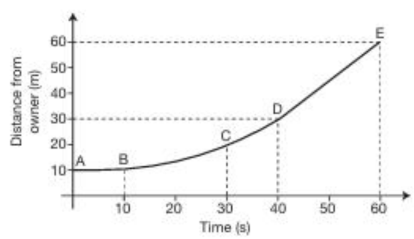
Because the tangent is judged by eye, different people may get different answers for the gradient. The answers given are calculated using a different technique which you will learn in a later unit, so don't expect your answers to be exactly the same as those given.

KEY POINTS

- To estimate the gradient of a curve at a point
  - draw the best estimate of the tangent at the point
  - find the gradient of this tangent.
- Be careful finding the rise and run when the scales on the axes are different.

EXAMPLE 2

A dog is running in a straight line away from its owner.
Part of the distance–time graph describing the motion is shown.



a Describe how the dog's speed varies.
b Estimate the dog's speed after 30 seconds.

Note: Remember that the gradient of a distance–time graph gives the speed.

a A to B: The gradient is zero, so the speed is zero. The dog is stationary for the first 10 seconds, 10 metres away from its owner.

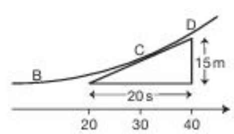
B to D: The gradient is gradually increasing, so the speed is gradually increasing. For the next 30 seconds the dog runs with increasing speed.

D to E: The gradient is constant and equal to $\frac{30}{20}$ or 1.5, so the dog is running at a constant speed of 1.5 m/s.

b Draw a tangent at C and calculate the gradient of the tangent.

The gradient is $\frac{15\text{ m}}{20\text{ s}} = 0.75$ m/s

so the speed of the dog is approximately 0.75 m/s.



EXAMPLE 3

The area of weed covering part of a pond doubles every 10 years. The area now covered is $100\,m^2$.

a Given that the area of weed, $A\,m^2$, after n years, is given by $A = 100 \times 2^{0.1n}$, draw the graph of A against n for $0 \le n \le 40$.

b By drawing suitable tangents, find the rate of growth of weed in m^2 per year after 10 years and after 30 years.

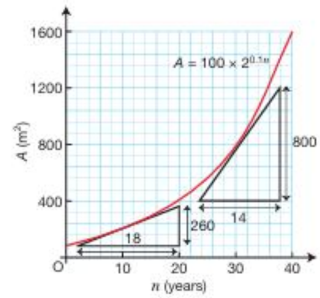
a

n (years)	0	10	20	30	40
A (m^2)	100	200	400	800	1600

b Rate of growth at 10 years $= \dfrac{260\ m^2}{18\ years} \simeq 14\,m^2/year$

Rate of growth at 30 years $= \dfrac{800\ m^2}{14\ years} \simeq 57\,m^2/year$

The rate of growth is clearly increasing with time.



TRANSLATING GRAPHS

EXAMPLE 4

SKILLS ANALYSIS

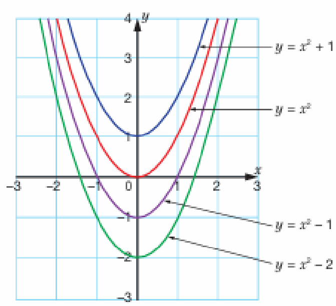
The diagram shows the graphs of $y = x^2$, $y = x^2 - 1$, $y = x^2 - 2$ and $y = x^2 + 1$

Describe the transformation of the graph of

a $y = x^2$ to $y = x^2 - 1$

b $y = x^2$ to $y = x^2 - 2$

c $y = x^2$ to $y = x^2 + 1$



The transformation from $y = x^2$ to the other three graphs is a vertical translation.

a $y = x^2$ to $y = x^2 - 1$ is a translation of $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

b $y = x^2$ to $y = x^2 - 2$ is a translation of $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$

c $y = x^2$ to $y = x^2 + 1$ is a translation of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

EXAMPLE 5

SKILLS ANALYSIS

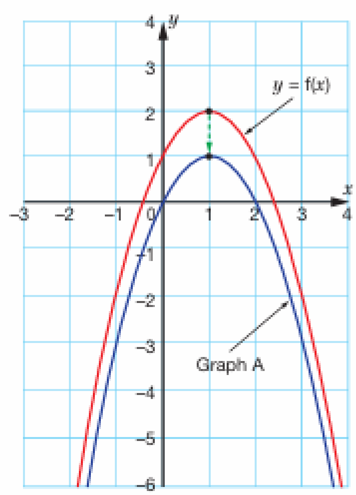
Graph A is a translation of $y = f(x)$

a Describe the translation.

b Find the image of the **maximum point** (1, 2).

c Write down the equation of graph A.

d If $f(x) = -x^2 + 2x + 1$, find the equation of graph A.



a A translation of $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

b $(1, 2) \rightarrow (1, 1)$

c $y = f(x) - 1$

d $y = (-x^2 + 2x + 1) - 1 = -x^2 + 2x$

EXAMPLE 6

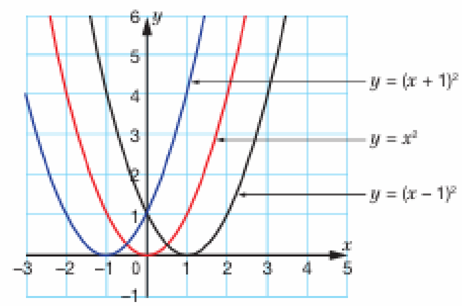
SKILLS
ANALYSIS

The diagram shows the graphs of $y = x^2$, $y = (x - 1)^2$ and $y = (x + 1)^2$

Describe the transformation of the graph of

a $y = x^2$ to $y = (x - 1)^2$

b $y = x^2$ to $y = (x + 1)^2$



The transformation from $y = x^2$ to the other three graphs is a horizontal translation.

a $y = x^2$ to $y = (x - 1)^2$ is a translation of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

b $y = x^2$ to $y = (x + 1)^2$ is a translation of $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

EXAMPLE 7

SKILLS
ANALYSIS

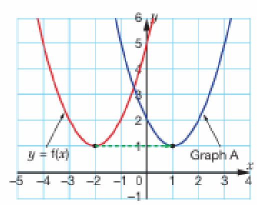
Graph A is a translation of $y = f(x)$

a Describe the translation.

b Find the image of the **minimum point** (–2, 1).

c Write down the equation of graph A.

d If $f(x) = x^2 + 4x + 5$, find the equation of graph A.



a A translation of $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$

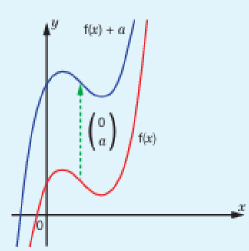
b (1, 1)

c $y = f(x - 3)$

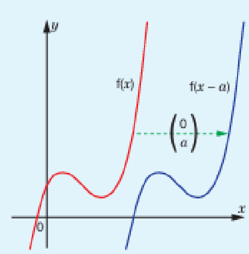
d $y = (x - 3)^2 + 4(x - 3) + 5 = x^2 - 2x + 2$

KEY POINTS

- The graph of $y = f(x) + a$ is a translation of the graph of $y = f(x)$ by $\begin{pmatrix} 0 \\ a \end{pmatrix}$



- The graph of $y = f(x - a)$ is a translation of the graph of $y = f(x)$ by $\begin{pmatrix} a \\ 0 \end{pmatrix}$
- The graph of $y = f(x + a)$ is a translation of the graph of $y = f(x)$ by $\begin{pmatrix} -a \\ 0 \end{pmatrix}$
- Be very careful with the signs, they are the opposite to what most people expect.



REFLECTING GRAPHS

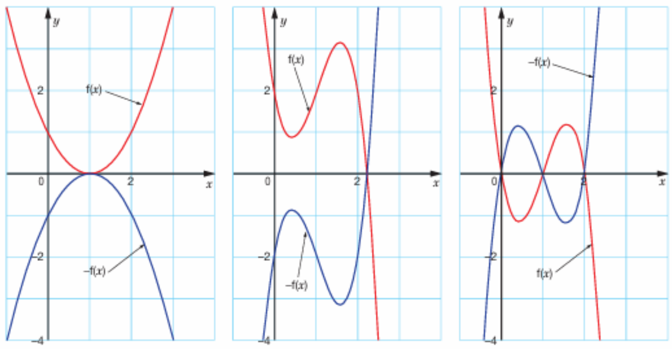
REFLECTION IN THE x-AXIS

EXAMPLE 8

SKILLS

ANALYSIS

The three graphs below show $y = f(x)$ and $y = -f(x)$ plotted on the same axes.
In each case the transformation is a reflection in the x-axis.



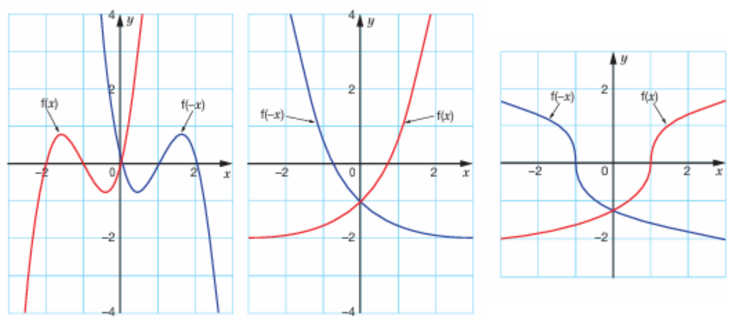
REFLECTION IN THE y-AXIS

EXAMPLE 9

SKILLS

ANALYSIS

The three graphs below show $y = f(x)$ and $y = f(-x)$ plotted on the same axes.
In each case the transformation is a reflection in the y-axis.



KEY POINTS

- The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.
- The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the y-axis.

STRETCHING GRAPHS

STRETCHES PARALLEL TO THE y-AXIS

A graph can be stretched or compressed in the y direction by multiplying the function by a number.

EXAMPLE 10

SKILLS

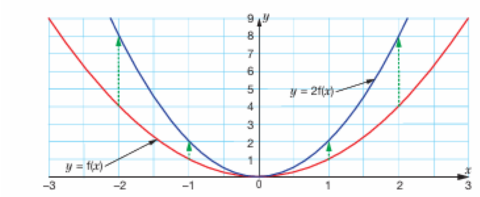
ANALYSIS

$f(x) = x^2$

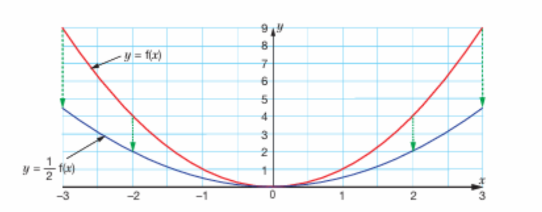
a Draw $y = f(x)$ and $y = 2f(x)$ on the same axes and describe the transformation.

b Draw $y = f(x)$ and $y = \frac{1}{2}f(x)$ on the same axes and describe the transformation.

a The arrows on the graph show that the transformation is a stretch of 2 in the y direction.



b The arrows on the graph show that the transformation is a stretch of $\frac{1}{2}$ in the y direction.



EXAMPLE 11

SKILLS

ANALYSIS

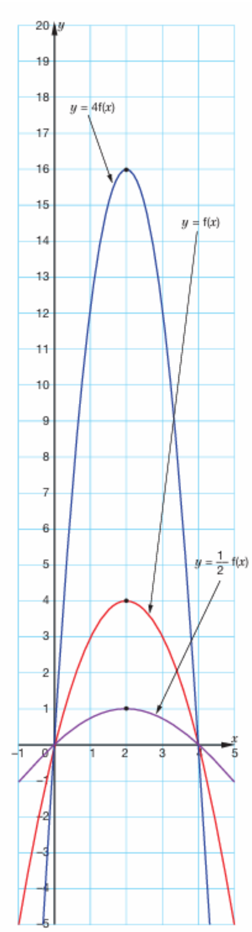
The diagram shows the graph of $y = f(x)$ which has a maximum point at (2, 4).

a Find the maximum point of $y = 4f(x)$

b Find the maximum point of $y = \frac{1}{4}f(x)$

a The graph of $y = 4f(x)$ is a stretch of the graph $y = f(x)$ in the y direction by a **scale factor** of 4, so the maximum point is (2, 16).

b The graph of $y = \frac{1}{4}f(x)$ is a stretch of the graph $y = f(x)$ in the y direction by a scale factor of $\frac{1}{4}$, so the maximum point is (2, 1).

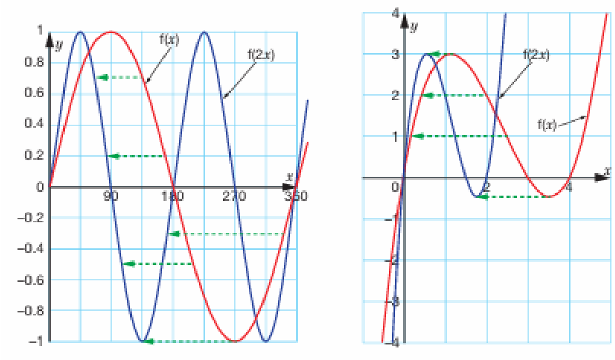


STRETCHES PARALLEL TO THE x-AXIS

EXAMPLE 12
SKILLS
ANALYSIS

The two graphs below show $y = f(x)$ and the $y = f(2x)$ plotted on the same axes.

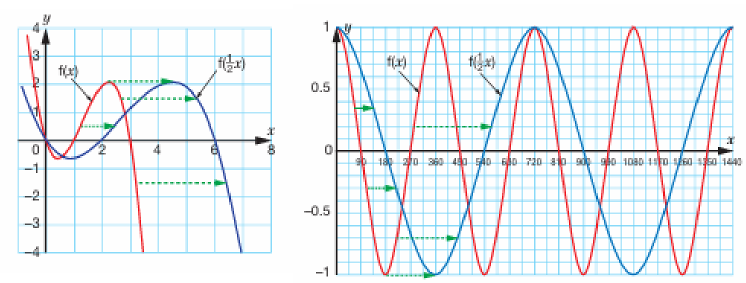
In each case the transformation from $y = f(x)$ to $y = f(2x)$ is a stretch scale factor $\frac{1}{2}$ parallel to the x-axis.



EXAMPLE 13
SKILLS
ANALYSIS

The two graphs below show a function $f(x)$ and the function $f(\frac{1}{2}x)$ plotted on the same axes.

In each case the transformation from $y = f(x)$ to $y = f(\frac{1}{2}x)$ is a stretch scale factor 2 parallel to the x-axis.

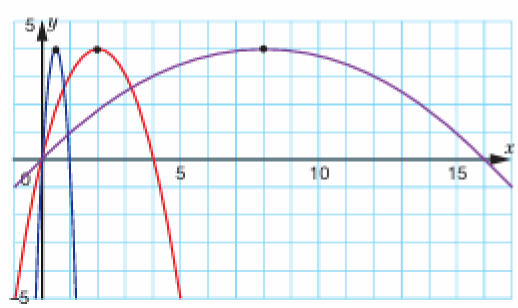


EXAMPLE 14
SKILLS
ANALYSIS

The diagram shows the graph of $y = f(x)$ (red curve) which has a maximum point at (2, 4).

a Find the maximum point of $y = f(4x)$

b Find the maximum point of $y = f(\frac{1}{4}x)$

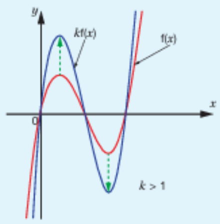


a The graph of $y = f(4x)$ is a stretch of the graph $y = f(x)$ in the x direction by a scale factor of $\frac{1}{4}$, so the maximum point is $(\frac{1}{2}, 4)$.

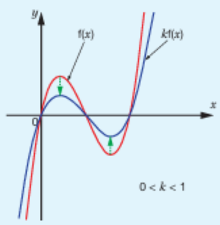
b The graph of $y = f(\frac{1}{4}x)$ is a stretch of the graph $y = f(x)$ in the x direction by a scale factor of 4, so the maximum point is (8, 4).

KEY POINTS

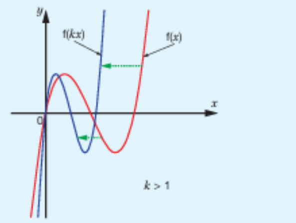
- The graph of $y = kf(x)$ is a stretch of the graph $y = f(x)$ with a scale factor of k parallel to the y-axis (all y-co-ordinates are multiplied by k).
  - If $k > 1$ the graph is stretched by k.

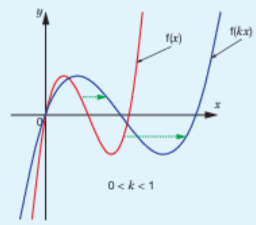


  - If $0 < k < 1$ the graph is compressed by k.



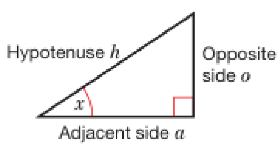
- The graph of $y = f(kx)$ is a stretch of the graph $y = f(x)$ with a scale factor of $\frac{1}{k}$ parallel to the x-axis (all x-co-ordinates are multiplied by $\frac{1}{k}$).
  - If $k > 1$ the graph is compressed by $\frac{1}{k}$
  - If $0 < k < 1$ the graph is stretched by $\frac{1}{k}$

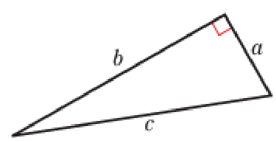




SHAPE AND SPACE 9

BASIC PRINCIPLES



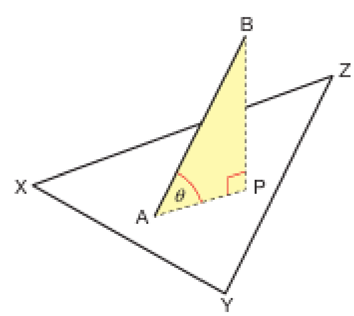


- Trig ratios:
  $\tan x = \frac{\text{opp}}{\text{adj}}$ $\sin x = \frac{\text{opp}}{\text{hyp}}$ $\cos x = \frac{\text{adj}}{\text{hyp}}$
- Identify the **hypotenuse**. This is the longest side: the side opposite the **right angle**. Then the opposite side is opposite the angle. And the **adjacent** side is adjacent to (next to) the angle.
- Pythagoras' Theorem: $a^2 + b^2 = c^2$

3D TRIGONOMETRY

SOLVING PROBLEMS IN 3D

The angle between a line and a plane is identified by dropping a perpendicular line from a point on the line onto the plane and by joining the point of contact to the point where the line intersects the plane. In the diagram, XYZ is a plane and AB is a line that meets the plane at an angle θ. BP is perpendicular from the top of the line, B, to the plane. P is the point of contact. Join AP. The angle θ, between the line and the plane, is angle BAP in the right-angled triangle ABP.

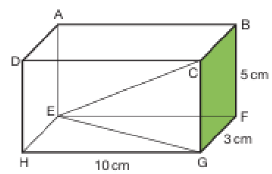


EXAMPLE 1 **SKILLS** ANALYSIS

ABCDEFGH is a **cuboid**.

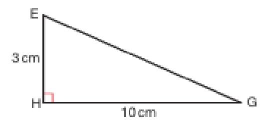
Find to 3 **significant figures**

a length EG
b length CE
c the angle CE makes with plane EFGH (angle CEG).



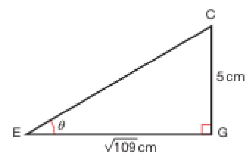
a Draw triangle EGH.

Using Pythagoras' Theorem
$EG^2 = 3^2 + 10^2 = 109$
$EG = \sqrt{109}$
EG = 10.4 cm (3 s.f.)



b Draw triangle CEG.

Using Pythagoras' Theorem
$CE^2 = 5^2 + 109 = 134$
$CE = \sqrt{134}$
CE = 11.6 cm (3 s.f.)



c Let angle CEG = θ

$\tan\theta = \frac{5}{\sqrt{109}} \Rightarrow \theta = \text{angle CEG} = 25.6°$ (3 s.f.)

KEY POINTS

When solving problems in 3D:

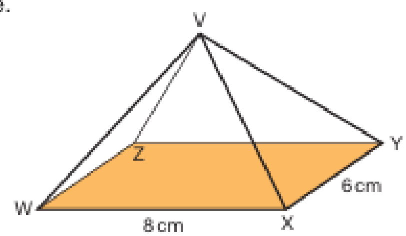
- Draw clear, large diagrams including all the facts.
- Redraw the appropriate triangle (usually right-angled) including all the facts. This simplifies a 3D problem into a 2D problem using Pythagoras' Theorem and trigonometry to solve for angles and lengths.
- Use all the **decimal places** shown on your calculator at each stage in your working to avoid errors in your final answer caused by **rounding** too soon.

EXAMPLE 2 **SKILLS** ANALYSIS

VWXYZ is a solid regular pyramid on a rectangular base WXYZ. WX = 8 cm and XY = 6 cm.
The **vertex** V is 12 cm vertically above the centre of the base.

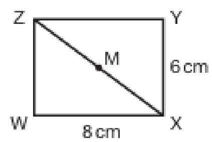
Find

a VX
b the angle between VX and the base WXYZ (angle VXZ)
c the area of pyramid **face** VWX.



a Let M be the mid-point of ZX.

Draw WXYZ in 2D.



By Pythagoras' Theorem on triangle ZWX

$ZX^2 = 6^2 + 8^2 = 100$

$ZX = 10\text{ cm} \Rightarrow MX = 5\text{ cm}$

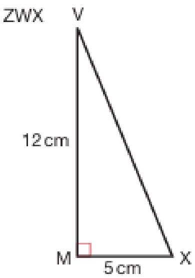
Draw triangle VMX.

V is vertically above the mid-point M of the base.

By Pythagoras' Theorem on triangle VMX

$VX^2 = 5^2 + 12^2 = 169$

$VX = 13\text{ cm}$



b Angle VXZ = Angle VXM = θ

$\tan\theta = \frac{12}{5} \Rightarrow \theta = 67.4°$ (3 s.f.)

$\Rightarrow$ Angle VXZ = 67.4° (3 s.f.)

c Let N be the **mid-point** of WX.

$\text{Area of triangle VWX} = \frac{1}{2} \times \text{base} \times \text{perpendicular height}$

$= \frac{1}{2} \times WX \times VN$

$= \frac{1}{2} \times 8 \times VN$

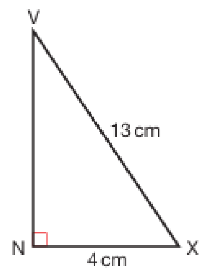
Draw triangle VNX in 2D.

By Pythagoras' Theorem on triangle VNX

$13^2 = 4^2 + VN^2$

$VN^2 = 13^2 - 4^2 = 153$

$VN = \sqrt{153} \Rightarrow \text{Area of VWX} = \frac{1}{2} \times 8 \times \sqrt{153} = 49.5\text{ cm}^2$ (3 s.f.)



HANDLING DATA 6

BASIC PRINCIPLES

- Draw and interpret **bar charts** and frequency diagrams (for equal class intervals).
- Work out the width of class intervals.
- Work out the mid-point of class intervals.
- Write down the **modal class**, and the interval that contains the **median** from a grouped frequency table.
- Estimate the **range** and work out an estimate for the **mean** from a grouped frequency table.

DRAWING HISTOGRAMS

Histograms appear similar to bar charts, but there are clear differences. Bar charts have frequency on the vertical axis and the frequency equals the height of the bar. Histograms have frequency density on the vertical axis, which makes the frequency proportional to the area of the bar.

When data is presented in groups of different class widths (such as $0 \le x < 2$, $2 \le x < 10$ etc.) a histogram is drawn to display the information.

EXAMPLE 1

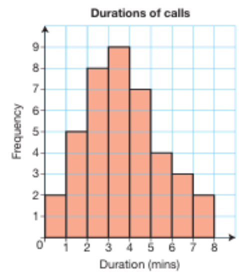
SKILLS

ANALYSIS

Ella records the time of 40 phone calls. The results are shown in the table on the next page.

Show the results on a bar chart.

TIME, t (mins)	FREQUENCY f
$0 \le t < 1$	2
$1 \le t < 2$	5
$2 \le t < 3$	8
$3 \le t < 4$	9
$4 \le t < 5$	7
$5 \le t < 6$	4
$6 \le t < 7$	3
$7 \le t < 8$	2

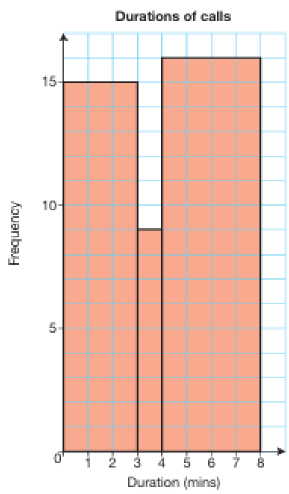


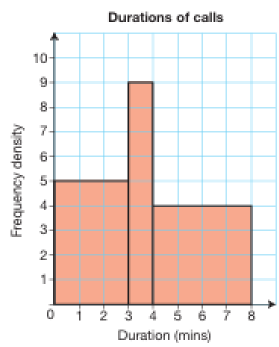
HINT

The frequency densities are worked out in a calculation table. Four columns are needed.

Ella decides to group the same results as shown in the following table.
Show the results on a bar chart and on a histogram.

TIME, t (mins)	FREQUENCY	CLASS WIDTH	FREQUENCY DENSITY
$0 \le t < 3$	15	3	$15 \div 3 = 5$
$3 \le t < 4$	9	1	$9 \div 1 = 9$
$4 \le t < 8$	16	4	$16 \div 4 = 4$





The bar chart displays the **frequencies**.

The histogram displays the **frequency densities**.

The bar chart with groups of different widths gives a very misleading impression of the original data.

The histogram gives a good impression of the original data although some of the fine detail has been lost in the process of grouping.

KEY POINTS

- For data grouped in unequal class intervals, you need a histogram.
- In a histogram, the area of the bar represents the frequency.
- The height of each bar is the frequency density.
- Frequency density $= \dfrac{\text{frequency}}{\text{class width}}$

Data with different class intervals is often used to estimate information such as means, proportions and probabilities.

EXAMPLE 2
SKILLS
INTERPRETATION

Use Ella's grouped data from Example 1 to

a calculate an estimate of the mean time of a phone call

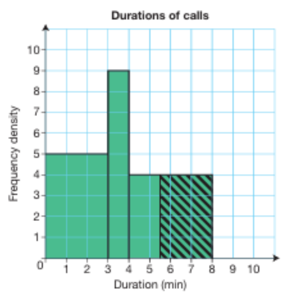
b estimate the number of phone calls that are more than $5\frac{1}{2}$ minutes.

HINT
Extend the calculation table to include the mid-point (x) of each group and fx for each group.

a

TIME, t (mins)	FREQUENCY, f	WIDTH	FREQUENCY DENSITY	MID-POINT, x	fx
$0 \le t < 3$	15	3	$15 \div 3 = 5$	1.5	22.5
$3 \le t < 4$	9	1	$9 \div 1 = 9$	3.5	31.5
$4 \le t < 8$	16	4	$16 \div 4 = 4$	6	96
	$\sum f = 40$				$\sum fx = 150$

Estimate of the mean $= \frac{\sum fx}{\sum f} = \frac{150}{40} = 3.75$ mins



The shaded area on the histogram represents the calls that are more than $5\frac{1}{2}$ minutes. This area is $4 \times 2.5 = 10$. As the area represents frequency, an estimate of the number of phone calls over $5\frac{1}{2}$ minutes is 10.

Note: The original ungrouped data suggests that only seven calls were more than $5\frac{1}{2}$ minutes, showing that accuracy is lost when data is grouped.

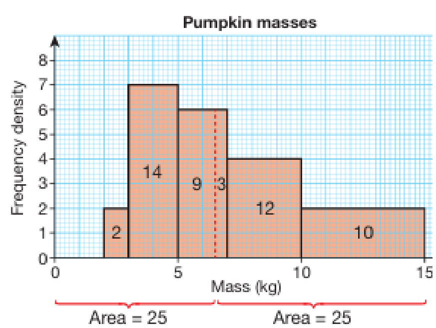
INTERPRETING HISTOGRAMS

Making accurate and logical conclusions from histograms is commonly required after they have been drawn.

EXAMPLE 3
SKILLS
INTERPRETATION

The histogram shows the **masses** of pumpkins in a farm shop.

Work out an estimate for the median mass.



Total frequency = total areas of all the bars

$= 1 \times 2 + 2 \times 7 + 2 \times 6 + 3 \times 4 + 5 \times 2 = 50$

The median is the 25.5th value and lies in the class $5 < m \le 7$

Frequency = area = 9, frequency density $= \frac{\text{frequency}}{\text{class width}} = 6$

Class width $= \frac{\text{frequency}}{\text{frequency density}} = 9 \div 6 = 1.5$

An estimate for the median is found by adding the class width to the lower class boundary: $5 + 1.5 = 6.5$ kg

SKILLS
INTERPRETATION

ACTIVITY 1

Comparison of data: Brain training

A company wishes to investigate how effective their brain training programme is. They first test how long it takes to solve a logic puzzle, then they allow the participants to use the programme for a week, and then they test the participants again.

Two sets of data were gathered from the same group of adults. The data shows the time taken to solve a puzzle before and after they have taken the course of 'Brain Training'.

TIME TAKEN (t seconds)	FREQUENCY (BEFORE), f_1	FREQUENCY (AFTER), f_2
$12 \le t < 18$	10	13
$18 \le t < 21$	9	14
$21 \le t < 24$	16	16
$24 \le t < 27$	10	14
$27 \le t < 33$	15	3

Study the data carefully and use it to draw two separate histograms.

Use statistical methods to decide if this sample shows whether or not the 'Brain Training' programme makes a difference to a person's ability to solve the puzzle.

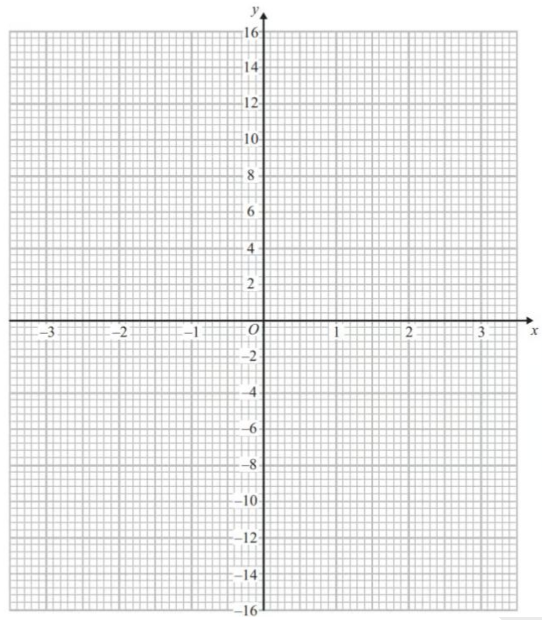
Revision questions

1.

Complete the table of values for $y = x^3 - 4x$

x	-3	-2	-1	0	1	2	3
y			3	0			15

On the grid, draw the graph of $y = x^3 - 4x$ from $x = -3$ to $x = 3$

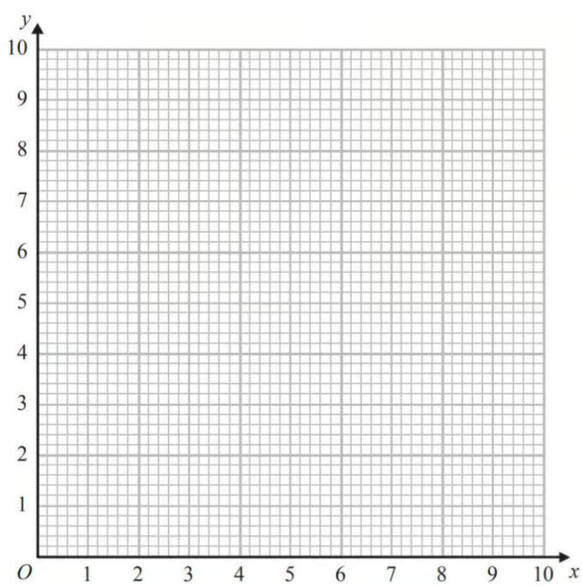


2.

Complete the table of values for $y = \frac{4}{x}$

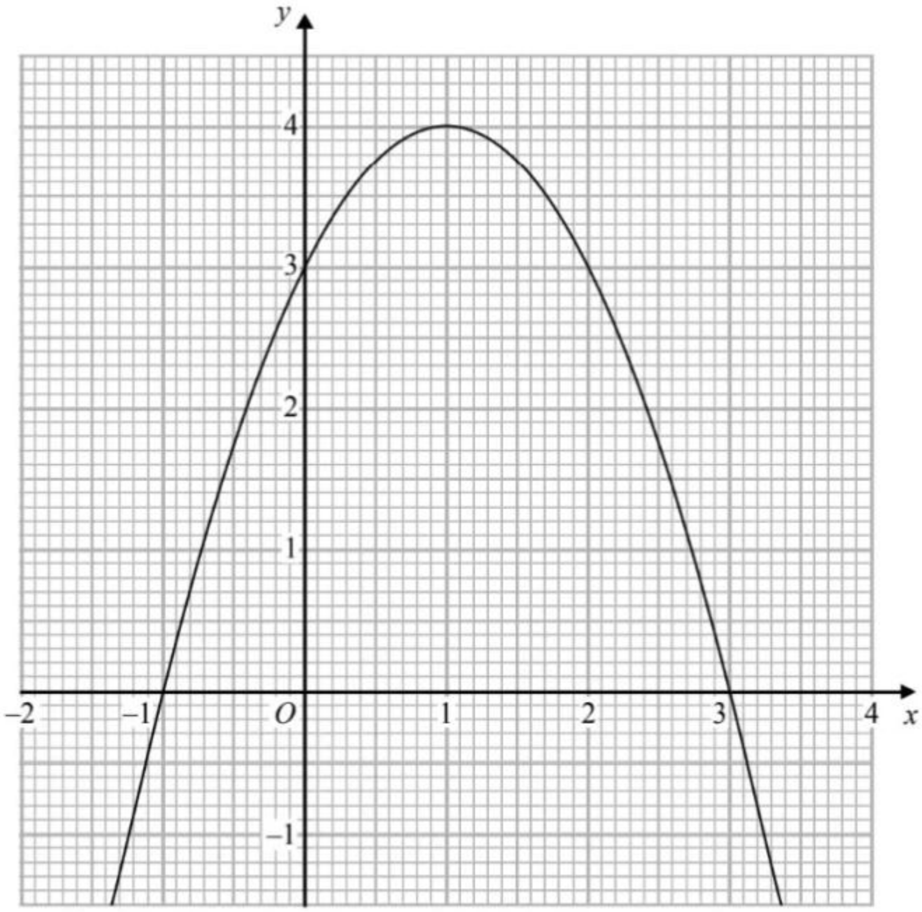
x	0.5	1	2	4	5	8
y		4	2			

On the grid, draw the graph of $y = \frac{4}{x}$ for $0.5 \leqslant x \leqslant 8$



3.

The graph of $y = f(x)$ is drawn on the grid.



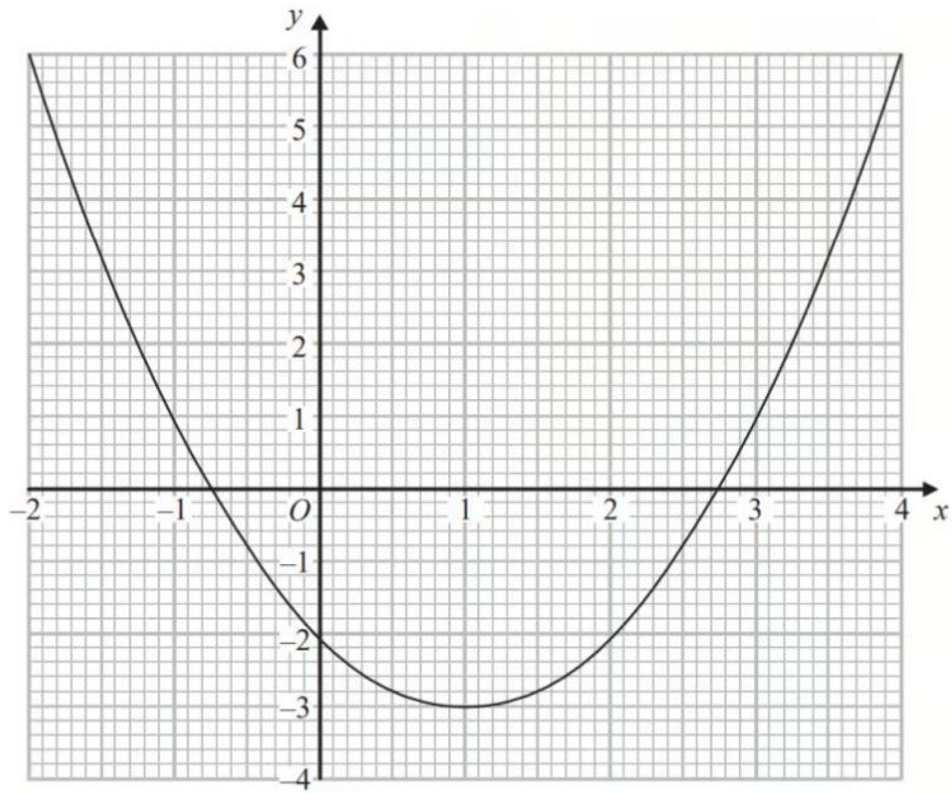
Write down the coordinates of the turning point of the graph.

Write down the roots of $f(x) = 2$

Write down the value of $f(0.5)$

4.

The graph of $y = f(x)$ is drawn on the grid.



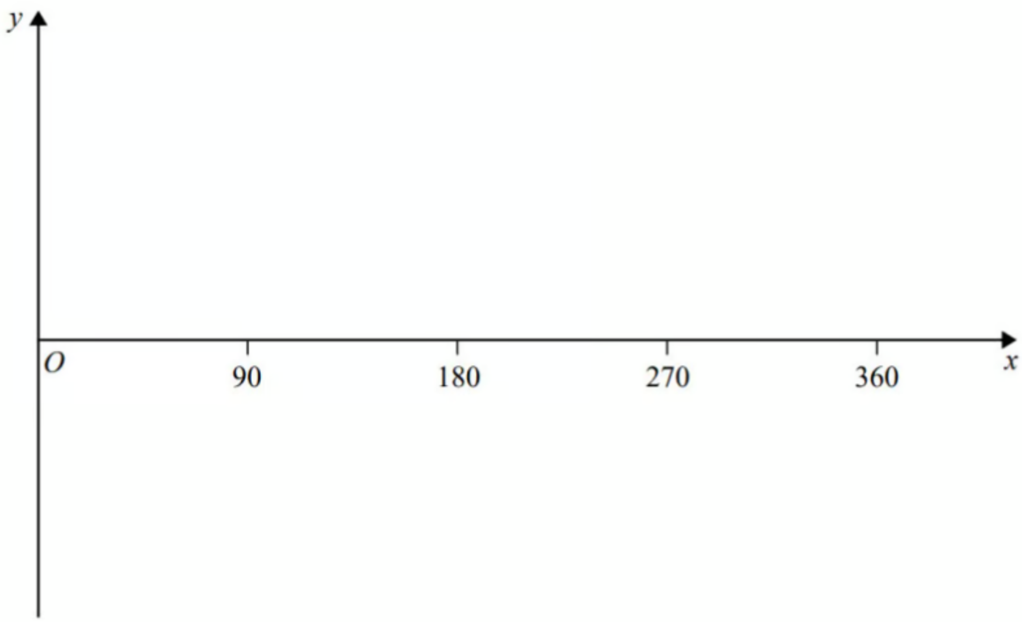
Write down the coordinates of the turning point of the graph.

Write down estimates for the roots of $f(x) = 0$

Use the graph to find an estimate for $f(1.5)$

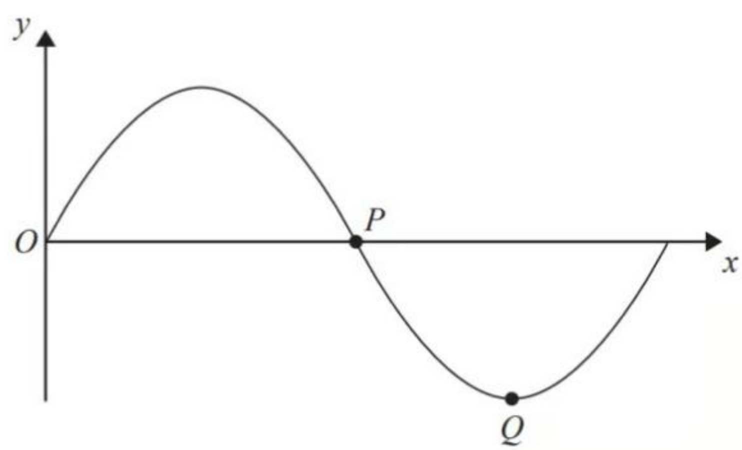
5.

Sketch the graph of $y = \cos x^{\circ}$ for $0 \leqslant x \leqslant 360$



6.

The diagram shows part of a sketch of the curve $y = \sin x^{\circ}$.



Write down the coordinates of the point P.

Write down the coordinates of the point Q

7.