Edexcel

OL IGCSE

Mathematics

CODE: (4CP0)

Unit 06

4MB1

Number 06

LEARNING OBJECTIVES

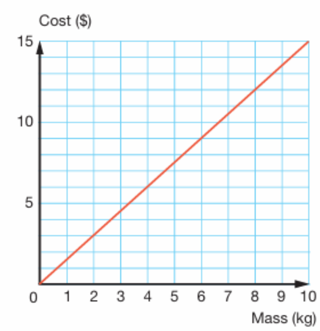
- Recognise and use direct proportion
- Recognise and use inverse proportion
- Use index laws to simplify numerical expressions involving negative and fractional indices

BASIC PRINCIPLES

- Plot curved graphs.
- Make a table of values, plotting the points and joining them using a smooth curve.
- Rules of **indices**:
  - When multiplying, add the indices: $x^m \times x^n = x^{m+n}$
  - When dividing, subtract the indices: $x^m \div x^n = x^{m-n}$
  - When raising to a **power**, multiply the indices: $(x^m)^n = x^{mn}$
- Express numbers in **prime factor** form: $72 = 8 \times 9 = 2^3 \times 3^2$
- Convert metric units of length.
- Understand **ratio**.

DIRECT PROPORTION

If two quantities are in direct proportion, then when one is multiplied or divided by a number, so is the other.
For example, if 4 kg of apples cost \$6, then 8 kg cost \$12, 2kg cost \$3 and so on.



This relationship produces a straight-line graph through the origin.
When two quantities are in direct proportion, the graph of the relationship will always be a straight line through the origin.

EXAMPLE 1

SKILLS PROBLEM SOLVING

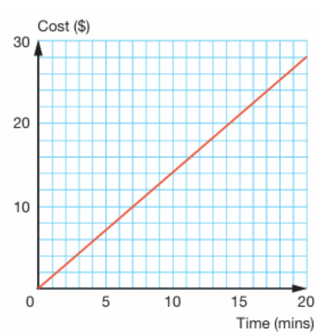
The cost of a phone call is directly proportional to its length. A three-minute call costs \$4.20.

a What is the cost of an eight-minute call? **b** A call costs \$23.10. How long is it?

a 3 minutes costs \$4.20
$\Rightarrow$ 1 minute costs $\$4.20 \div 3 = \$1.40$
$\Rightarrow$ 8 minutes costs $8 \times \$1.40 = \$11.20$

b 1 minute costs \$1.40
$\Rightarrow$ \$1 gives $\frac{1}{1.40}$ minutes
$\Rightarrow$ \$23.10 gives $23.10 \times \frac{1}{1.40}$ minutes $= 16.5$ minutes

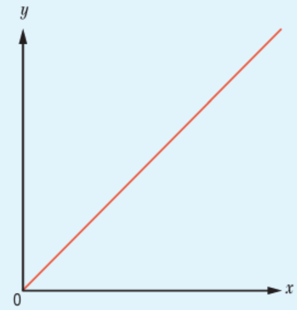
In Example 1, the graph of cost plotted against the number of minutes is a straight line through the origin.



KEY POINTS

If two quantities are in direct proportion:

- when one is multiplied or divided by a number, so is the other
- their ratio stays the same as they increase or decrease
- the graph of the relationship will always be a straight line through the origin.



INVERSE PROPORTION

ACTIVITY 1

SKILLS
MODELLING

Ethan and Mia are planning a journey of 120 km. They first work out the time it will take them travelling at various speeds.

Copy and complete the table showing their results.

TIME, t (hours)	2	3	4	5		8
SPEED, v (km/hr)		40			20	
$t \times v$		120				

Plot their results on a graph of v km/hr against t hours for $0 \le t \le 8$

Use your graph to find the speed required to do the journey in $2\frac{1}{2}$ hours.

Is there an easier way to find this speed?

The quantities (time and speed) in Activity 1 are in **inverse proportion**.

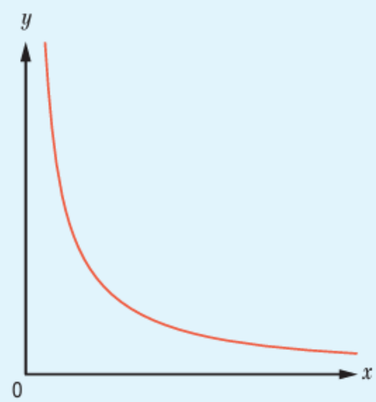
The graph you drew in Activity 1 is called a **reciprocal graph**.

In Activity 1 when the two quantities are multiplied together the answer is always 120. The **product** of two quantities is always constant if the quantities are in inverse proportion. This makes it easy to calculate values.

KEY POINTS

When two quantities are in inverse proportion:

- the graph of the relationship is a reciprocal graph
- one quantity increases at the same **rate** as the other quantity decreases, for example, as one doubles (× 2) the other halves (÷ 2)
- their product is constant.



FRACTIONAL INDICES

The laws of indices can be extended to fractional indices.

ACTIVITY 3

SKILLS
REASONING

a Use your calculator to work out:

i $4^{\frac{1}{2}}$ **ii** $9^{\frac{1}{2}}$ **iii** $16^{\frac{1}{2}}$ **iv** $25^{\frac{1}{2}}$ **v** $36^{\frac{1}{2}}$

What does this suggest $a^{\frac{1}{2}}$ means?

Check your theory with some other numbers.

b Use your calculator to work out:

i $8^{\frac{1}{3}}$ **ii** $27^{\frac{1}{3}}$ **iii** $64^{\frac{1}{3}}$

What does this suggest $a^{\frac{1}{3}}$ means?

Check your theory with some other numbers.

Using the laws of indices, $4^{\frac{1}{2}} \times 4^{\frac{1}{2}} = 4^{\left(\frac{1}{2}+\frac{1}{2}\right)} = 4^1 = 4$

But $\sqrt{4} \times \sqrt{4} = 4$

This means that $4^{\frac{1}{2}} = \sqrt{4} = 2$ so $4^{\frac{1}{2}}$ means the square root of 4

Similarly, $8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} = 8^{\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)} = 8^1 = 8$

But $\sqrt[3]{8} \times \sqrt[3]{8} \times \sqrt[3]{8} = 8$

This means that $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$ so $8^{\frac{1}{3}}$ means the cube root of 8

In a similar way it can be shown that $5^{\frac{1}{4}} = \sqrt[4]{5}$, $7^{\frac{1}{5}} = \sqrt[5]{7}$ and so on.

Writing numbers as powers can simplify the **working**.

KEY POINTS

- Use the rules:
  - $x^{\frac{1}{m}} = \sqrt[m]{x}$
  - $x^{\frac{n}{m}} = \left(\sqrt[m]{x}\right)^n = \sqrt[m]{x^n}$
- Or, write the numbers in prime factor form first.

NEGATIVE INDICES

The rules of indices can also be extended to work with negative indices.

In an earlier number chapter you used $10^{-2} = \frac{1}{10^2}$, $10^{-3} = \frac{1}{10^3}$ and so on.

This also works with numbers other than 10.

EXAMPLE 7

Show that $2^{-2} = \frac{1}{2^2}$

$2^2 \div 2^4 = 2^{-2}$ (Using $x^m \div x^n = x^{m-n}$)

But $2^2 \div 2^4 = \frac{2^2}{2^4} = \frac{1}{2^2} \Rightarrow 2^{-2} = \frac{1}{2^2}$

In a similar way it can be shown that $2^{-3} = \frac{1}{2^3}$, $5^{-4} = \frac{1}{5^4}$ and so on.

Note that $2^{-1} = \frac{1}{2^1} = \frac{1}{2}$

EXAMPLE 11

Show that $2^0 = 1$

$2^3 \div 2^3 = 2^0$ (Using $x^m \div x^n = x^{m-n}$)

But $2^3 \div 2^3 = \frac{2^3}{2^3} = 1 \Rightarrow 2^0 = 1$

In a similar way it can be shown that $3^0 = 1$, $4^0 = 1$, and so on.

KEY POINTS

- $x^{-n} = \frac{1}{x^n}$ for any number n, $x \neq 0$
- $\left(\frac{x}{y}\right)^{-n} = \frac{1}{\left(\frac{x}{y}\right)^n} = \left(\frac{y}{x}\right)^n$, $x, y \neq 0$
- $x^{-1} = \frac{1}{x}$, $x \neq 0$
- $x^0 = 1$, $x \neq 0$

Algebra 06

BASIC PRINCIPLES

- Use formulae relating one **variable** to another, for example $A = \pi r^2$
- Substitute numerical values into formulae and relate the answers to applied real situations.
- Derive simple formulae (linear, quadratic and cubic).
- Know and use the laws of **indices**:
  $a^m \times a^n = a^{m+n}$
  $a^m \div a^n = a^{m-n}$
  $(a^m)^n = a^{mn}$
- Know and use the laws of indices involving fractional, negative and zero **powers** in a number context.

PROPORTION

If two quantities are related to each other, given enough information, it is possible to write a formula describing this relationship.

SKILLS
ANALYSIS

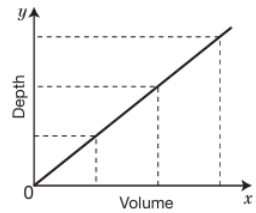
ACTIVITY 1

Copy and complete this table to show which paired items are related.

VARIABLES	RELATED? (YES OR NO)
Area of a circle A and its **radius** r	Y
Circumference of a circle C and its radius r	
Distance travelled, D, at a constant speed, x	
Mathematical ability, M, and a person's height, h	
Cost of a tin of paint, P, and its density, d	
Weight of water, W, and its volume, v	
Value of a painting, V, and its area, a	
Swimming speed, S, and collar size, x	

DIRECT PROPORTION-LINEAR

When water is poured into an empty cubical fish tank, each litre that is poured in increases the depth by a fixed amount.



A graph of depth, y, against volume, x, is a straight line through the origin, showing a linear relationship.
In this case, y is directly proportional to x. If y is doubled, so is x. If y is halved, so is x, and so on.

This relationship can be expressed in any of these ways:
y is directly proportional to x.
y varies directly with x.
y varies as x.

All these statements have the same meaning.
In symbols, direct proportion relationships can be written as y xx. The x sign can then be replaced by '= k' to give the formula y = kx, where k is the constant of proportionality.
The graph of y = kx is the equation of a straight line through the origin, with gradient k.

EXAMPLE 1

SKILLS
PROBLEM SOLVING

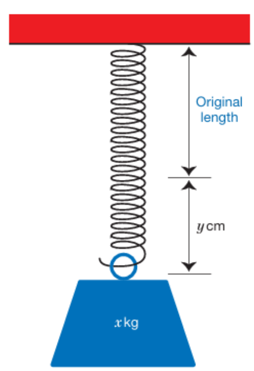
The extension, y cm, of a spring is directly proportional to the **mass**, x kg, hanging from it.

If $y = 12$ cm when $x = 3$ kg, find

a the formula for y in terms of x

b the extension y cm when a 7 kg mass is attached

c the mass x kg that produces a 20 cm extension.

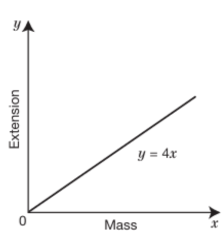


a y is directly proportional to x, so $y \propto x \Rightarrow y = kx$
$y = 12$ when $x = 3$, so $12 = k \times 3$
therefore, $k = 4$
Hence the formula is $y = 4x$

b When $x = 7$, $y = 4 \times 7 = 28$
The extension produced from a 7 kg mass is 28 cm.

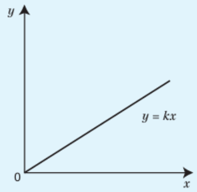
c When $y = 20$ cm, $20 = 4x$
therefore, $x = 5$

A 5 kg mass produces a 20 cm extension.
The graph of extension plotted against mass would look like this.



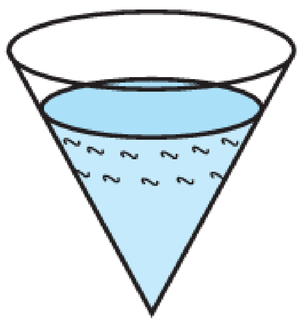
KEY POINTS

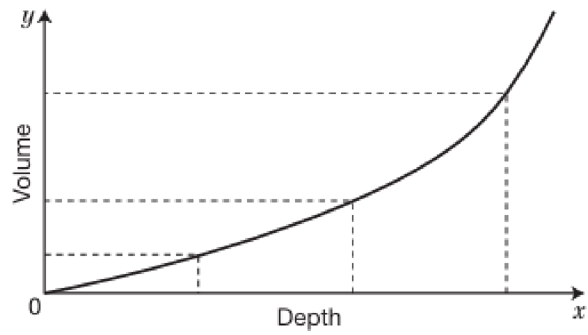
- y is directly proportional to x is written as $y \propto x$ and this means $y = kx$, for some constant k.
- The graph of y against x is a straight line through the origin.



DIRECT PROPORTION - NONLINEAR

Water is poured into an empty inverted cone. Each litre poured in will result in a different depth increase. A graph of volume, y, against depth, x, will illustrate a direct nonlinear relationship.





This is the graph of a **cubic** relationship.
This relationship can be expressed in *either* of these ways:

- y is directly proportional to x cubed.
- y varies as x cubed.

Both statements have the same meaning.
In symbols, this relationship is written as $y \propto x^3$
The $\propto$ sign can then be replaced by '$= k$' to give the formula $y = kx^3$, where k is the constant of proportionality.
A quantity can also be directly proportional to the square or the square root of another quantity.

EXAMPLE 2

SKILLS
ANALYSIS

Express these relationships as equations with constants of proportionality.

a y is directly proportional to x squared.
b m varies directly with the cube of n.
c s is directly proportional to the square root of t.
d v squared varies as the cube of w.

a $y \propto x^2 \Rightarrow y = kx^2$
b $m \propto n^3 \Rightarrow m = kn^3$
c $s \propto \sqrt{t} \Rightarrow s = k\sqrt{t}$
d $v^2 \propto w^3 \Rightarrow v^2 = kw^3$

KEY POINTS

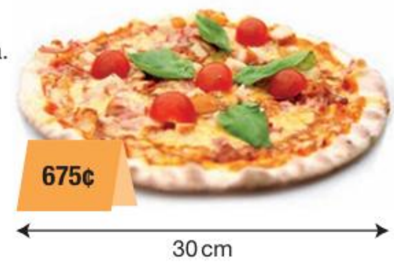
- If y is proportional to x squared then $y \propto x^2$ and $y = kx^2$, for some constant k.
- If y is proportional to x cubed then $y \propto x^3$ and $y = kx^3$, for some constant k.
- If y is proportional to the square root of x then $y \propto \sqrt{x}$ and $y = k\sqrt{x}$, for some constant k.

EXAMPLE 3

SKILLS

PROBLEM SOLVING

The cost of Luciano's take-away pizzas (C cents) is directly proportional to the square of the **diameter** (d cm) of the pizza. A 30 cm pizza costs 675 cents.



a Find a formula for C in terms of d and use it to find the price of a 20 cm pizza.

b What size of pizza can you expect for \$4.50?

a C is proportional to d^2, so $C \propto d^2$ — $C = kd^2$

$C = 675$ when $d = 30$ — $675 = k(30)^2$

$k = 0.75$

The formula is therefore $C = 0.75d^2$

When $d = 20$ — $C = 0.75(20)^2$

$C = 300$

The cost of a 20 cm pizza is 300 cents (\$3).

b When $C = 450$ — $450 = 0.75d^2$

$d^2 = 600$

$d = \sqrt{600} = 24.5$ (3 s.f.)

A \$4.50 pizza should be 24.5 cm in diameter.

INVERSE PROPORTION

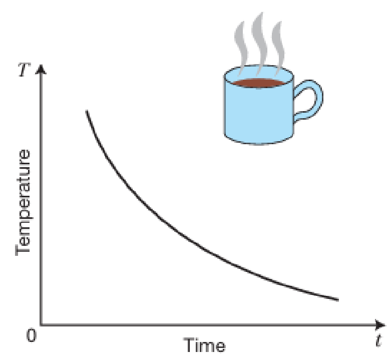
The temperature of a cup of coffee decreases as time increases.

A graph of temperature (T) against time (t) shows an **inverse** relationship.

This can be expressed as: 'T is inversely proportional to t'.

In symbols, this is written as $T \propto \frac{1}{t}$

The $\propto$ sign can then be replaced by '$= k$' to give the formula $T = \frac{k}{t}$ where k is the constant of proportionality.



EXAMPLE 4

SKILLS

ANALYSIS

Express these equations as relationships with constants of proportionality.

a y is inversely proportional to x squared.

b m varies inversely as the cube of n.

c s is inversely proportional to the square root of t.

d v squared varies inversely as the cube of w.

a $y \propto \frac{1}{x^2} \Rightarrow y = \frac{k}{x^2}$

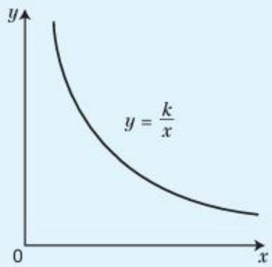
b $m \propto \frac{1}{n^3} \Rightarrow m = \frac{k}{n^3}$

c $s \propto \frac{1}{\sqrt{t}} \Rightarrow s = \frac{k}{\sqrt{t}}$

d $v^2 \propto \frac{1}{w^3} \Rightarrow v^2 = \frac{k}{w^3}$

KEY POINT

- If y is inversely proportional to x then $y \propto \frac{1}{x}$ and $y = \frac{k}{x}$, for some constant k. The graph of y plotted against x looks like this.



EXAMPLE 5

SKILLS

PROBLEM SOLVING

Sound intensity, I dB (decibels), is inversely proportional to the square of the distance, d m, from the source. At a music festival, it is 110 dB, 3 m away from a speaker.

a Find the formula relating I and d.

b Calculate the sound intensity 2 m away from the speaker.

c At what distance away from the speakers is the sound intensity 50 dB?

a I is inversely proportional to d^2 $\quad I = \frac{k}{d^2}$

$I = 110$ when $d = 3$ $\quad 110 = \frac{k}{3^2}$

$k = 990$

The formula is therefore $I = \frac{990}{d^2}$

b When $d = 2$ $\quad I = \frac{990}{2^2} = 247.5$

The sound intensity is 247.5 dB, 2 m away. (This is enough to cause deafness.)

c When $I = 50$ $\quad 50 = \frac{990}{d^2}$

$d^2 = 19.8$

$d = 4.45$ (3 s.f.)

The sound intensity is 50 dB, 4.45 m away from the speakers.

INDICES

NEGATIVE INDICES AND FRACTIONAL INDICES INCLUDING ZERO

The laws of indices can also be applied to algebraic expressions with fractional or negative powers.

EXAMPLE 6

Work out the value of

a $27^{\frac{2}{3}}$

b $16^{-\frac{3}{4}}$

a $27^{\frac{2}{3}} = \left(27^{\frac{1}{3}}\right)^2 = 3^2 = 9$

HINT
Use the rule $(x^m)^n = x^{mn}$. Work out the cube root of 27 first. Then square the answer.

b $16^{-\frac{3}{4}} = \dfrac{1}{\left(16^{\frac{1}{4}}\right)^3} = \dfrac{1}{2^3} = \dfrac{1}{8}$

HINT
Use $x^{-n} = \dfrac{1}{x^n}$

EXAMPLE 7

Simplify

a $a^{\frac{1}{2}} \times a^{\frac{1}{2}} \times a$

b $3a^2 \times 2a^{-3}$

c $(6a^{-2}) \div (2a^2)$

d $\left(\dfrac{a^2}{b^4}\right)^{-\frac{1}{2}}$

a $a^{\frac{1}{2}} \times a^{\frac{1}{2}} \times a = a^{\left(\frac{1}{2}+\frac{1}{2}+1\right)} = a^2$

b $3a^2 \times 2a^{-3} = 6 \times a^{2+(-3)} = 6a^{-1} = \dfrac{6}{a}$

HINT
An expression containing a negative power is not fully simplified: x^{-1} should be written as $\dfrac{1}{x}$

c $(6a^{-2}) \div (2a^2) = 3a^{(-2-2)} = 3a^{-4} = \dfrac{3}{a^4}$

d $\left(\dfrac{a^2}{b^4}\right)^{-\frac{1}{2}} = \dfrac{1}{\left(\dfrac{a^2}{b^4}\right)^{\frac{1}{2}}} = \left(\dfrac{b^4}{a^2}\right)^{\frac{1}{2}} = \dfrac{b^2}{a}$

KEY POINTS

- $a^{\frac{1}{m}} = \sqrt[m]{a}$
- $a^{\frac{n}{m}} = \left(\sqrt[m]{a}\right)^n = \sqrt[m]{a^n}$
- $a^0 = 1$, $a \neq 0$
- $a^{-1} = \dfrac{1}{a}$, $a \neq 0$
- $a^{-n} = \dfrac{1}{a^n}$ for any number n, $a \neq 0$
- $\left(\dfrac{a}{b}\right)^{-n} = \dfrac{1}{\left(\dfrac{a}{b}\right)^n} = \left(\dfrac{b}{a}\right)^n$ for any number n, a, $b \neq 0$

Sequences

BASIC PRINCIPLES

- These are examples of important sequences:

  Natural numbers: 1, 2, 3, 4, …

  Even numbers: 0, 2, 4, 6, …

  Odd numbers: 1, 3, 5, 7, …

  Triangle numbers: 1, 3, 6, 10, …

  Square numbers: 1, 4, 9, 16, … (The squares of the natural numbers)

  Powers of 2: 1, 2, 4, 8, … (Numbers of the form 2^n)

  Powers of 10: 1, 10, 100, … (Numbers of the form 10^n)

  Prime numbers: 2, 3, 5, 7, … (Note that 1 is not a prime number)
- Solve two-step linear equations.
- Solve linear **inequalities**.

CONTINUING SEQUENCES

A set of numbers that follows a definite pattern is called a sequence. You can continue a sequence if you know how the terms are related.

ACTIVITY 1

SKILLS
PROBLEM SOLVING

Seema is decorating the walls of a hall with balloons in preparation for a disco. She wants to place the balloons in a triangular pattern.

To make a 'triangle' with one row she needs one balloon.

To make a triangle with two rows she needs three balloons.

To make a triangle with three rows she needs six balloons.

The number of balloons needed form the sequence 1, 3, 6, …
Describe in words how to continue the sequence and find the next three terms.

Seema thinks she can work out how many balloons are needed for n rows by using the formula $\frac{1}{2}n(n+1)$. Her friend Julia thinks the formula is $\frac{1}{2}n(n-1)$.

Who is correct?

If Seema has 100 balloons find, using 'trial and improvement', how many rows she can make, and how many balloons will be left over (remaining).

FORMULAE FOR SEQUENCES

Sometimes the sequence is given by a formula. This means that any term can be found without working out all the previous terms.

EXAMPLE 3

SKILLS ANALYSIS

Find the first four terms of the sequence given by the nth term = $2n - 1$

Find the 100th term.

Substituting $n = 1$ into the formula gives the first term as	$2 \times 1 - 1 = 1$
Substituting $n = 2$ into the formula gives the second term as	$2 \times 2 - 1 = 3$
Substituting $n = 3$ into the formula gives the third term as	$2 \times 3 - 1 = 5$
Substituting $n = 4$ into the formula gives the fourth term as	$2 \times 4 - 1 = 7$
Substituting $n = 100$ into the formula gives the 100th term as	$2 \times 100 - 1 = 199$

EXAMPLE 4

SKILLS ANALYSIS

A sequence is given by the nth term = $4n + 2$. Find the value of n for which the nth term equals 50.

$4n + 2 = 50 \Rightarrow 4n = 48 \Rightarrow n = 12$

So the 12th term equals 50.

KEY POINT

- When a sequence is given by a formula, any term can be worked out.

ACTIVITY 2

SKILLS ANALYSIS

HINT
Try nth term = $2n$, nth term = $3n$, …

A sequence is given by nth term = an. What is the connection between a and the numbers in the sequence?

A sequence is given by nth term = $3n + 2$. Copy and fill in the table.

n	1	2	3	4	10
$3n + 2$	5				

What is the connection between the numbers 3 and 2 and the numbers in the sequence?

A sequence is given by nth term = $an + b$

What is the connection between a and b and the numbers in the sequence?

THE DIFFERENCE METHOD

When it is difficult to spot a pattern in a sequence, the difference method can often help. Under the sequence write down the differences between each pair of terms. If the differences show a pattern then the sequence can be extended.

EXAMPLE 5

SKILLS ANALYSIS

Find the next three terms in the sequence 2, 5, 10, 17, 26, …

Sequence:	2		5		10		17		26
Differences:		3		5		7		9	

= 5 − 2

= 10 − 5

The differences increase by 2 each time so the table can now be extended.

										= 26 + 11		= 37 + 13			
Sequence:	2		5		10		17		26		**37**		**50**		**65**
Differences:		3		5		7		9		**11**		**13**		**15**	

If the pattern in the differences is not clear, add a third row giving the differences between the terms in the second row. More rows can be inserted until a pattern is found but remember not all sequences will result in a pattern.

EXAMPLE 6

SKILLS
ANALYSIS

Find the next three terms in the sequence 0, 3, 16, 45, 96, …

Sequence:	0		3		16		45		96
Differences:		3		13		29		51	
			10		16		22		
				6		6			

Now the table can be extended to give:

Sequence:	0		3		16		45		96		**175**		**288**		**441**
Differences:		3		13		29		51		**79**		**113**		**153**	
			10		16		22		**28**		**34**		**40**		
				6		6		**6**		**6**		**6**			

KEY POINT

- The difference method finds patterns in sequences when the patterns are not obvious.

ACTIVITY 3

SKILLS
PROBLEM SOLVING

Use a spreadsheet to find the next three terms of these sequences.

2, 4, 10, 20, 34, 52, …

3, 5, 10, 20, 37, 63, 100, …

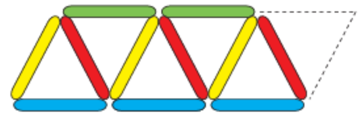
2, 1, 1, 0, –4, –13, –29, …

FINDING A FORMULA FOR A SEQUENCE

ACTIVITY 4

SKILLS
REASONING

Seema decides to decorate the walls of the hall with different patterns of cylindrical balloons. She starts with a triangular pattern.



Seema wants to work out how many balloons she will need to make 100 triangles.

If t is the number of triangles and b is the number of balloons, copy and fill in the following table.

t	1	2	3	4	5	6
b						

$t = 1$

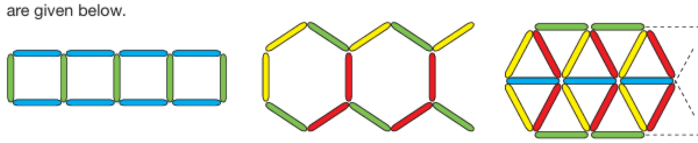
$t = 2$

As the sequence in the row headed b increases by 2 each time add another row to the table headed $2t$.

t	1	2	3	4	5	6
b						
$2t$	2	4	6	8	10	12

Write down the formula that connects b and $2t$. How many balloons does Seema need to make 100 triangles?

Use paper clips (or other small objects) to make up some other patterns that Seema might use and find a formula for the number of balloons needed. Some possible patterns are given below.

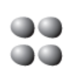


ACTIVITY 5

SKILLS
REASONING

12 is a square **perimeter** number because 12 stones can be arranged as the perimeter of a square.

The first square perimeter number is 4.



Copy and complete the following table where n is the square perimeter number and s is the number of stones needed.

n	1	2	3	4	5	6
s	4					

Use the method of Activity 4 to find a formula for the number of pebbles in the nth square perimeter number.

KEY POINT

- If the first row of differences is constant and equal to a then the formula for the nth term will be nth term $= an + b$ where b is another constant.

ARITHMETIC SEQUENCES

When the difference between any two consecutive terms in a sequence is the same, the sequence is called an arithmetic sequence. The difference between the terms is called the common difference. The common difference can be positive or negative.

2 5 8 11 14 … is an arithmetic sequence with a common difference of 3
1 2 3 4 5 … is an arithmetic sequence with a common difference of 1
5 3 1 −1 −3 … is an arithmetic sequence with a common difference of −2
2 3 5 7 11 … is NOT an arithmetic sequence.

EXAMPLE 9

SKILLS
ANALYSIS

Show that **a** and **b** are arithmetic sequences.

a 5, 8, 11, 14, … **b** 21, 19, 17, 15, …

a Sequence: 5 8 11 14
Differences: 3 3 3

As the differences are constant, the sequence is an arithmetic sequence with common difference 3.

b Sequence: 21 19 17 15
Differences: −2 −2 −2

As the differences are constant, the sequence is an arithmetic sequence with common difference −2.

NOTATION

The letter a is used for the first term, and the letter d is used for the common difference.

1st term	2nd term	3rd term	4th term	…	nth term
a	$a + d$	$a + 2d$	$a + 3d$	…	$a + (n - 1)d$

An arithmetic sequence is given by $a, a + d, a + 2d, a + 3d, ..., a + (n - 1)d$

KEY POINTS

In an arithmetic sequence:

- The first term is a
- The common difference is d
- The nth term is $a + (n - 1)d$

SUM OF AN ARITHMETIC SEQUENCE

ACTIVITY 6

SKILLS
PROBLEM SOLVING

In this activity you will discover how Gauss added up the numbers from 1 to 100.

A good strategy with mathematical problems is to start with a simpler version of the problem.

To add up the numbers from 1 to 5 write the series forwards and backwards and add.

$$\begin{array}{r} 1+2+3+4+5 \\ +\; 5+4+3+2+1 \\ \hline 6+6+6+6+6 \end{array}$$

Adding five lots of six is done by calculating $5 \times 6 = 30$

This is twice the series so $1 + 2 + 3 + 4 + 5 = \frac{30}{2} = 15$

Use this technique to show that $1 + 2 + 3 + \ldots + 9 + 10 = 55$

Show that the answer to Gauss' problem $1 + 2 + 3 + \ldots + 99 + 100$ is 5050

Find a formula to add $1 + 2 + 3 + \ldots + (n - 1) + n$. Check that your formula works with Gauss' problem.

Gauss was adding up an arithmetic sequence. This is called an **arithmetic series**.

$4 + 7 + 10 + 13 + \ldots$ is an arithmetic series because the terms are an arithmetic sequence.

The general terms in an arithmetic sequence are $a, a + d, a + 2d, a + 3d, \ldots, a + (n - 1)d$

The notation S_n is used to mean 'the **sum** to n terms'.

S_6 means the sum of the first six terms, S_9 means the sum of the first nine terms and so on.

The sum of an arithmetic series is

$S_n = a + (a + d) + (a + 2d) + (a + 3d) + \ldots + (a + (n - 1)d)$

$S_n = \frac{n}{2}[2a + (n - 1)d]$

The proof of this follows Example 14.

EXAMPLE 12
SKILLS
PROBLEM SOLVING

Note: This is Gauss' problem.

Show that $1 + 2 + 3 + \ldots + 99 + 100 = 5050$ using the formula

$S_n = \frac{n}{2}[2a + (n - 1)d]$

$a = 1$, $d = 1$ and $n = 100 \Rightarrow S_{100} = \frac{100}{2}[2 + (100 - 1)1] = 50 \times 101 = 5050$

EXAMPLE 13
SKILLS
PROBLEM SOLVING

Find $4 + 7 + 10 + 13 + \ldots + 151$

$a = 4$ and $d = 3$.

The nth term is $a + (n - 1)d \Rightarrow 4 + (n - 1) \times 3 = 151$

$$\begin{aligned} 4 + 3n - 3 &= 151 \\ 3n &= 150 \\ n &= 50 \end{aligned}$$

$S_{50} = \frac{50}{2}[2 \times 4 + (50 - 1)3] = 3875$

EXAMPLE 14
SKILLS
PROBLEM SOLVING

The first term of an arithmetic series is 5 and the 20th term is 81.

Find the sum of the first 30 terms.

$a = 5$

The 20th term is $5 + (20 - 1)d \Rightarrow 5 + (20 - 1)d = 81$

$$\begin{aligned} 5 + 19d &= 81 \\ 19d &= 76 \\ d &= 4 \end{aligned}$$

$S_{30} = \frac{30}{2}[2 \times 5 + (30 - 1)4] = 1890$

PROOF OF $S_n = \frac{n}{2}[2a + (n - 1)d]$

The proof uses the same method as Activity 6. The series is written forwards and backwards and then added.

$$
\begin{array}{lllllll}
S_n & = a & + \quad (a + d) & + \ldots + & (a + (n-2)d) & + & (a + (n-1)d) \\
S_n & = (a + (n-1)d) & + \quad (a + (n-2)d) & + \ldots + & (a + d) & + & a \\
\hline
2S_n & = [2a + (n-1)d] & + \quad [2a + (n-1)d] & + \ldots + & [2a + (n-1)d] & + & [2a + (n-1)d]
\end{array}
$$

There are n terms that are all the same in the square brackets.

$\Rightarrow 2S_n = n[2a + (n - 1)d]$

$\Rightarrow S_n = \frac{n}{2}[2a + (n - 1)d]$

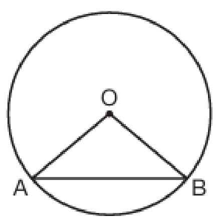
KEY POINTS

- $a + (a + d) + (a + 2d) + (a + 3d) + \ldots + (a + (n - 1)d)$ is an arithmetic series.
- The sum to n terms of an arithmetic series is $S_n = \frac{n}{2}[2a + (n - 1)d]$

Shape and space 6

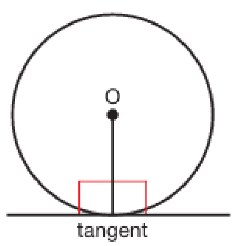
BASIC PRINCIPLES

- Triangle OAB is **isosceles**.

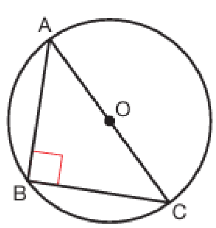


- A **tangent** is a straight line that touches a circle at one point only.

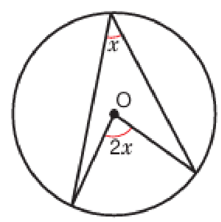
  The angle between a tangent and the **radius** is 90°

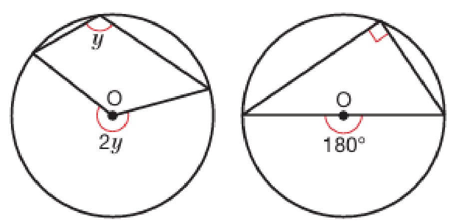


- An angle in a **semicircle** is always a **right angle**.



- The angle at the centre of a circle is twice the angle at the **circumference** when both are **subtended** by the same **arc**.



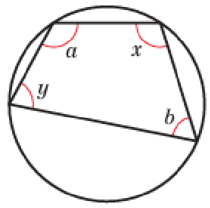


- A figure is **cyclic** if a circle can be drawn through its vertices. The vertices are **concyclic** points.

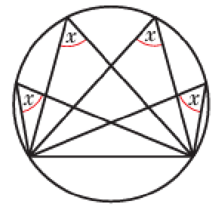
- Opposite angles of a **cyclic quadrilateral sum** to 180°.

  $a° + b° = 180°$

  $x° + y° = 180°$



- Angles in the same **segment** are equal.



CIRCLE THEOREMS 2

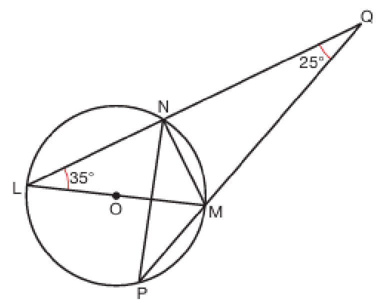
The basic circle theorems are frequently combined when solving circle geometry questions. These questions will allow you to revise your understanding of these basic circle theorems.

EXAMPLE 1

SKILLS
PROBLEM SOLVING

Find ∠PNM.

∠NPM = 35° (Angles in the same segment are equal)
∠LNM = 90° (Angle in a semicircle is 90°)
∠MNQ = 90° (Angles on a straight line total 180°)
So ∠PNM = 30° (Angle sum of ΔPNQ is 180°)

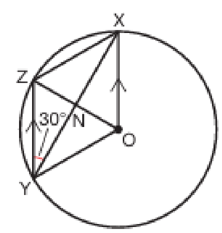


EXAMPLE 2

SKILLS
PROBLEM SOLVING

Prove that XY meets OZ at right angles.

∠OXY = 30° (**Alternate angle** to ∠ZYX)
∠OYX = 30° (Base angles in an isosceles triangle are equal)
∠YZO = 60° (Base angles in an isosceles triangle are equal)
∠ZNY = 90° (Angle sum of ΔZNY is 180°)



KEY POINTS

When trying to find angles or lengths in circles:

- Always draw a neat diagram, and include all the facts. Use a pair of **compasses** to draw all circles.
- Give a reason, in brackets, after each statement.

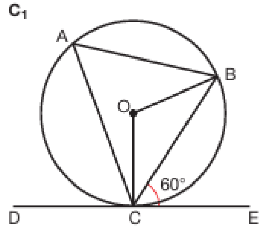
ALTERNATE SEGMENT THEOREM

ACTIVITY 1

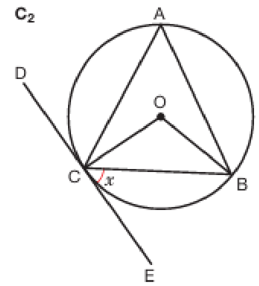
SKILLS
ANALYSIS

Find the angles in circle C_1 and circle C_2.

C_1



C_2



Copy and complete the table for C_1 and C_2.

CIRCLE	∠ECB	∠OCB	∠OBC	∠BOC	∠BAC
C_1	60°				
C_2	$x°$				

What do you notice about angles ECB and angles BAC?

The row for angles in C_2 gives the structure for a formal proof.

A full proof requires reasons for every stage of the calculation.

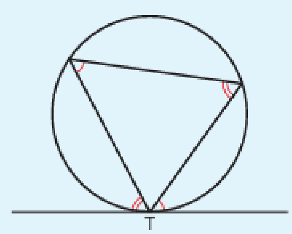
Calculations	**Reasons**
∠ECB = $x°$	General angle chosen.
∠OCB = $(90 - x)°$	Radius is perpendicular to tangent.
∠OBC = $(90 - x)°$	Base angles in an isosceles triangle are equal.
∠BOC = $2x°$	Angle sum of a triangle = 180°
∠BAC = $x°$	Angle at centre = 2 × angle at circumference.
∠BCE = ∠BAC	

Note: A formal proof of the theorem is not required by the specification.

KEY POINT

- The angle between a chord and a tangent is equal to the angle in the alternate segment.

  This is called the 'Alternate Segment Theorem'.



EXAMPLE 3

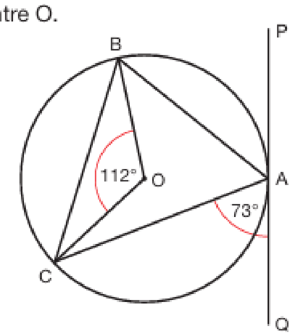
In the diagram, PAQ is the tangent at A to a circle with centre O.

Angle BOC = 112°

Angle CAQ = 73°

Work out the size of angle OBA.

Show your **working**, giving reasons for any statements you make.



Not drawn accurately

∠OBC = 34° (ΔOBC is isosceles)

∠ABC = 73° (Alternate segment theorem)

∠OBA = 73° – 34° = 39°

INTERSECTING CHORDS THEOREMS

ACTIVITY 2

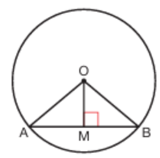
SKILLS
ANALYSIS

O is the centre of a circle.

OA and OB are radii. OM is perpendicular to AB.

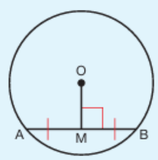
Prove that triangles OAM and OBM are **congruent** (the same).

Show that M is the mid-point of AB.



KEY POINTS

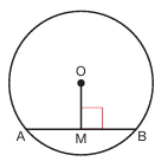
- A **chord** is a straight line connecting two points on a circle.
- The perpendicular from the centre of a circle to a chord bisects the chord and the line drawn from the centre of a circle to the mid-point of a chord is at right angles to the chord.



EXAMPLE 4

SKILLS
PROBLEM SOLVING

O is the centre of a circle.
The length of chord AB is 18 cm.
OM is perpendicular to AB.
Work out the length of AM.
State any circle theorems that you use.



AB = 18 cm

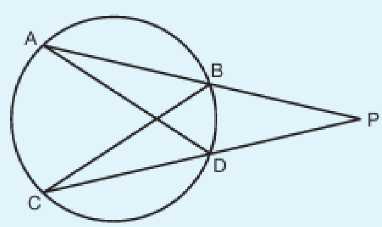
So, AM = $\frac{18}{2}$ = 9 cm (The perpendicular from the centre of a circle to a chord bisects the chord.

So, the length of AM will be exactly half the length of AB.)

KEY POINT

- Two chords intersecting outside a circle.

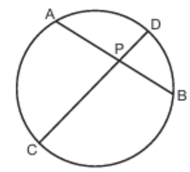
  AP × BP = CP × DP



EXAMPLE 5

SKILLS ANALYSIS

AP = 10, PD = 4 and PB = 6. Find CP.



If CP = x

AP × PB = CP × PD

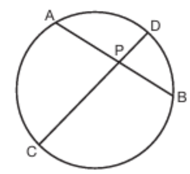
$10 \times 6 = x \times 4$

$60 = 4x$

$x = 15$

EXAMPLE 6

AP = 4, BP = 3 and CD = 8. Find DP.



If DP = x

CP = $8 - x$

AP × BP = CP × DP

$4 \times 3 = (8 - x)x$

$12 = 8x - x^2$

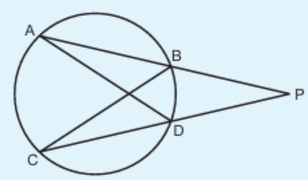
$x^2 - 8x + 12 = 0$

$(x - 2)(x - 6) = 0$

$x = 2$ or 6

KEY POINT

- Two chords intersecting outside a circle.
  AP × BP = CP × DP

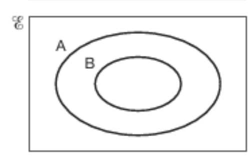


Sets 2

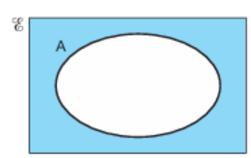
BASIC PRINCIPLES

- A **set** is a collection of objects, described by a list or a rule. — A = {1, 3, 5}
- Each object is an **element** or **member** of the set. — $1 \in A$, $2 \notin A$
- Sets are **equal** if they have exactly the same elements. — B = {5, 3, 1}, B = A
- The **number of elements** of set A is given by $n(A)$. — $n(A) = 3$
- The **empty set** is the set with no members. — { } or $\varnothing$
- The **universal set** contains all the elements being discussed in a particular problem. — $\mathscr{E}$

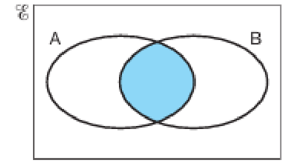
- B is a **subset** of A if every member of B is a member of A. — $B \subset A$



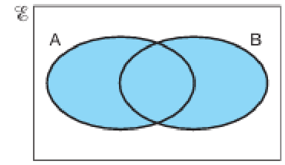
- The **complement** of set A is the set of all elements not in A. — A'



- The **intersection** of A and B is the set of elements which are in both A and B. $A \cap B$



- The **union** of A and B is the set of elements which are in A or B or both. $A \cup B$



EXAMPLE 1

SKILLS CRITICAL

THINKING

THREE-SET PROBLEMS

Questions involving three sets are more involved than two-set questions. The Venn diagram must show all the possible intersections of the sets.

There are 80 students studying either French (F), Italian (1) or Spanish (S) in a sixth form college. 7 of the students study all three languages. 15 study French and Italian, 26 study French and Spanish and 17 study Italian and Spanish. 43 study French and 52 study Spanish.

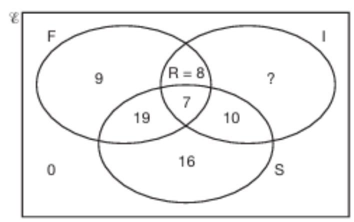
a Draw a Venn diagram to show this information. b How many study only Italian?

a Start by drawing three intersecting ovals labelled F, I and S.

As 7 students study all three subjects, put the number 7 in the intersection of all three circles.

15 students study French and Italian, so $n(F \cap I) = 15$
This number includes the 7 studying all three, so
$15 - 7 = 8$ must go in the region marked R.

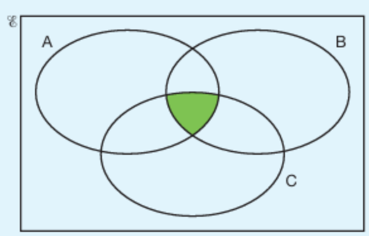
The other numbers are worked out in a similar way.
This shows, for example, that 19 students study French and Spanish only.



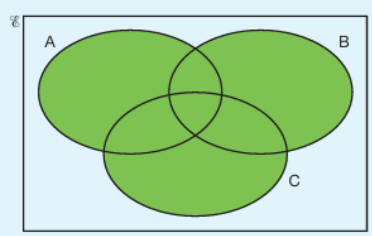
b As there are 80 students, the numbers must all add up to 80. The numbers shown add up to 69, so the number doing Italian only (the region marked '?') is $80 - 69 = 11$

KEY POINTS

- $A \cap B \cap C$ is the intersection of A, B and C i.e. where all three sets intersect.
- $A \cup B \cup C$ is the union of A, B and C, that is, all three sets combined.



$A \cap B \cap C$



$A \cup B \cup C$

PRACTICAL PROBLEMS

Entering the information from a problem into a Venn diagram usually means the numbers in the sets can be worked out. Sometimes it is easier to use some algebra as well.

EXAMPLE 2

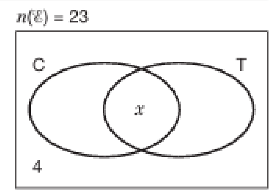
SKILLS

CRITICAL THINKING

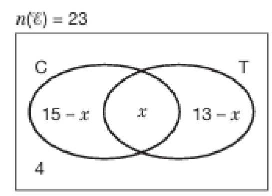
In a class of 23 students, 15 like coffee and 13 like tea. 4 students don't like either drink. How many like **a** both drinks, **b** tea only, **c** coffee only?

Enter the information into a Venn diagram in stages. Let C be the set of coffee drinkers and T the set of tea drinkers. Let x be the number of students who like both.

The 4 students who don't like either drink can be put in, along with x for the students who like both.



As 15 students like coffee, 15 – x students like coffee only and so 15 – x goes in the region shown. Similarly, 13 like tea so 13 – x goes in the region shown.



a The number who like coffee, tea or both is 23 – 4 = 19

This means $n(C \cup T) = 19$

So $(15 - x) + x + (13 - x) = 19$

$\Rightarrow 28 - x = 19$

$\Rightarrow x = 9$

So 9 students like both.

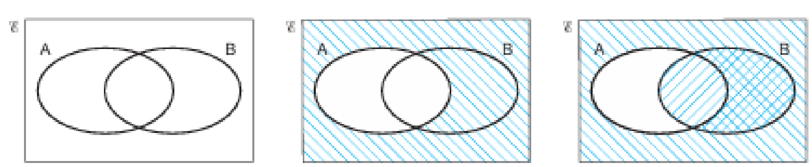
b The number who like tea only is $13 - x = 4$

c The number who like coffee only is $15 - x = 6$

SHADING SETS

Sometimes it can be difficult to find the intersection or union of sets in a Venn diagram. If one set is shaded in one direction and the other set in another direction, then the intersection is given wherever there is cross shading; the union is given by all the areas that are shaded.

The diagrams show first the sets A and B, then the set A′ shaded one way, then the set B shaded another way.



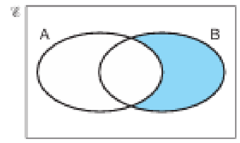
Sets A and B | Set A′ shaded one way | Set B shaded the other way

EXAMPLE 3

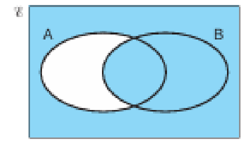
SKILLS
ANALYSIS

Show on a Venn diagram

a $A' \cap B$ **b** $A' \cup B$

a


Shading shows $A' \cap B$

b


Shading shows $A' \cup B$

KEY POINT

- Shading sets differently makes it easier to find the intersection or union.

ACTIVITY 1

SKILLS
ANALYSIS

This activity is about De Morgan's Laws. One law states that $(A \cup B)' = A' \cap B'$.

Shade copies of Diagram 4 to show the following sets: $A \cup B$, $(A \cup B)'$, A', B' and $A' \cap B'$ and therefore prove the law.

Another law states $(A \cap B)' = A' \cup B'$. Use copies of Diagram 4 to prove this law.

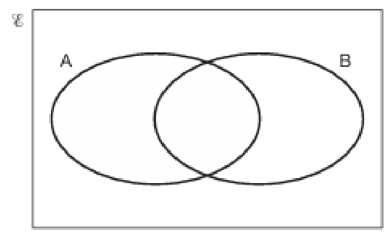


Diagram 4

SET-BUILDER NOTATION

Sets can be described using set-builder notation:

A = {x such that x > 2} means 'A is the set of all x such that x is greater than 2'.

Rather than write 'such that' the notation A = {x: x>2) is used.

B = {x: x > 2, x is positive integer) means the set of positive integers x such that x is greater than 2. This means B = {3, 4, 5, 6, ...).

C = {x: x < 2 or x > 2} can be written as {x: x <2}U {x: x > 2}.
The symbol U (union) means that the set includes all values satisfied by either inequality.

Certain sets of numbers are used so frequently that they are given special symbols.

■N is the set of natural numbers or positive integers {1, 2, 3, 4, ...).
■Z is the set of integers {..., -2, -1, 0, 1, 2, ...}.
■Q is the set of rational numbers. These are numbers that can be written as recurring or terminating decimals. Q does not contain numbers like √2 or л.
■R is the set of real numbers. This contains Q and numbers like √2 or or π.

EXAMPLE 4

SKILLS
ANALYSIS

Express in set-builder notation

a the set of natural numbers which are greater than 2
b the set of all real numbers greater than 2.

a $\{x: x > 2, x \in \mathbb{N}\}$
b $\{x: x > 2, x \in \mathbb{R}\}$

Note: a ≠ b since, for example, 3.2 is a real number but it is not a natural number.

EXAMPLE 5

List these sets.

a $\{x: x \text{ is even}, x \in \mathbb{N}\}$
b $\{x: x = 3y, y \in \mathbb{N}\}$

a $\{2, 4, 6, \ldots\}$
b $\{3, 6, 9, 12, \ldots\}$

Revision questions

1.

Yesterday it took 5 cleaners $4\frac{1}{2}$ hours to clean all the rooms in a hotel.

There are only 3 cleaners to clean all the rooms in the hotel today.

Each cleaner is paid £8.20 for each hour or part of an hour they work.

How much will each cleaner be paid today?

2.

Given that $y \propto \frac{1}{x^2}$, complete this table of values.

x	1	2	5	10
y				1

3.

The intensity of the sound, I watts/m^2, received from a loudspeaker is inversely proportional to the square of the distance, d metres, from the loudspeaker.

When $d = 2$, $I = 30$
Work out the value of I when $d = 10$

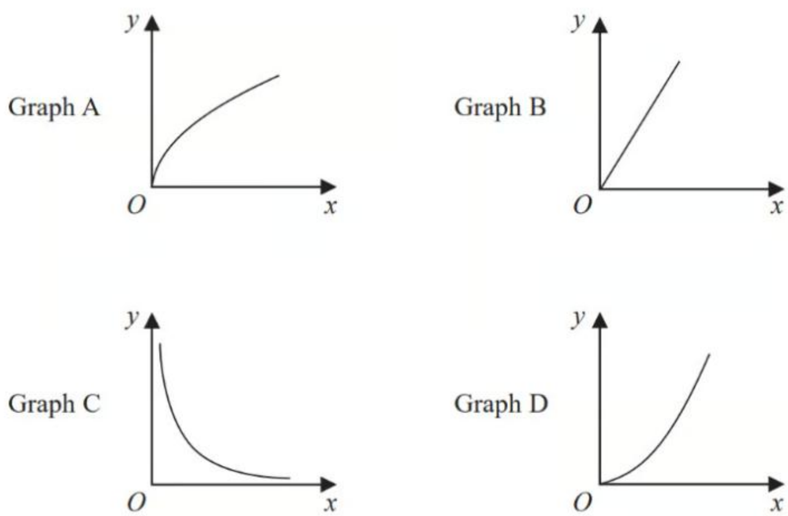
4.

T is inversely proportional to d^2

$T = 12$ when $d = 8$

Find the value of T when $d = 0.5$

5.



The graphs of y against x represent four different types of proportionality.

Match each type of proportionality in the table to the correct graph.

Type of proportionality	Graph letter
$y \propto x$	
$y \propto x^2$	
$y \propto \sqrt{x}$	
$y \propto \dfrac{1}{x}$	

6.

A company has to make a large number of boxes.

The company has 6 machines.
All the machines work at the same rate.
When all the machines are working, they can make all the boxes in 9 days.

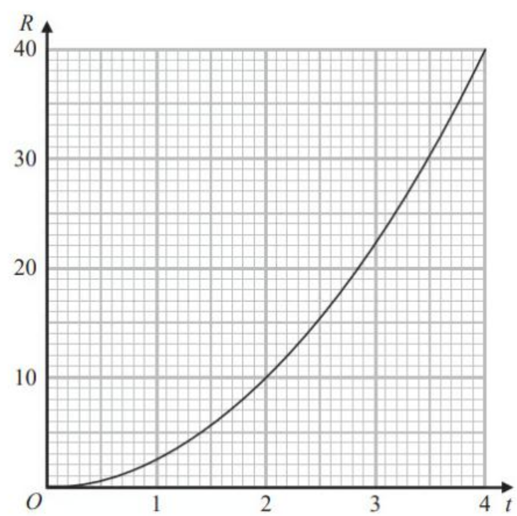
The table gives the number of machines working each day.

	day 1	day 2	day 3	all other days
Number of machines working	3	4	5	6

Work out the total number of days taken to make all the boxes.

7.

R is proportional to t^2
The graph shows the relationship between R and t for $0 \leqslant t \leqslant 4$



Find a formula for R in terms of t.

Given also that $R = \dfrac{8}{5x}$

show that t is inversely proportional to $\sqrt{x}$ for $t > 0$

8.

F is inversely proportional to the square of v.

Given that $F = 6.5$ when $v = 4$

find a formula for F in terms of v.

9.

y is directly proportional to $\sqrt[3]{x}$

$y = 1\frac{1}{6}$ when $x = 8$

Find the value of y when $x = 64$

10.

y is inversely proportional to d^2
When $d = 10, y = 4$

d is directly proportional to x^2
When $x = 2, d = 24$

Find a formula for y in terms of x.
Give your answer in its simplest form.